

THE SPACE OF COHEN–MACAULAY CURVES

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ABSTRACT. One can consider the Hilbert scheme as a natural compactification of the space of smooth projective curves with fixed Hilbert polynomial. Here we consider a different modular compactification, namely the functor CM parameterizing curves together with a finite map to \mathbb{P}^n that is generically a closed immersion.

We prove that CM is an algebraic space by constructing a scheme W and a representable, surjective and smooth map $\pi: W \rightarrow CM$. Moreover, we show that CM satisfies the valuative criterion for properness.

1. INTRODUCTION

A classical problem in algebraic geometry is to find and to describe moduli spaces of different geometric objects. The objects considered here are embedded projective *curves*, that is, one-dimensional subschemes of a given projective space that do not have embedded or isolated points. Moreover, we assume that the degree d and the genus g of the curve are fixed, that is, that the curve has a given Hilbert polynomial $p(t) = dt + 1 - g$.

A well-known compactification of the space of all curves in a given projective space \mathbb{P}^n is the *Hilbert scheme*. For a fixed polynomial $p(t)$, the Hilbert functor \mathcal{Hilb}^p parameterizes flat families of closed subschemes of \mathbb{P}^n with Hilbert polynomial $p(t)$. Grothendieck defined this functor in [Gro61c] and proved that it is represented by a projective scheme Hilb^p . The Hilbert scheme compactification of the space of embedded curves is obtained by parameterizing not only curves but all *subschemes* having Hilbert polynomial $p(t)$. For example, the Hilbert scheme Hilb^{3t+1} of twisted cubics in \mathbb{P}^3 contains not only the smooth twisted cubic curves but also plane curves with embedded or isolated points.

The moduli space of curves without embedded or isolated points that we are interested in is represented by an open subscheme of the Hilbert scheme, and we seek a compactification that avoids the degenerated schemes mentioned above.

Hønsen proposed in [Høn05] the following modular compactification of the space of curves. Instead of looking at curves with an embedding into the given projective space, he considered curves with a finite

map to the projective space that is generically a closed immersion. Concretely, this means that the map is an isomorphism onto its image away from a finite set of closed points. For example, the normalization of a plane nodal curve in \mathbb{P}^3 gives a finite map to \mathbb{P}^3 that is an isomorphism onto the image away from the singularity.

Hønsen proved that the moduli space CM of such pairs (C, i) , where $i: C \rightarrow \mathbb{P}^n$ denotes the finite map, is an algebraic space by verifying the conditions in *Artin's criteria* for representability [Art69, Theorem 3.4]. Furthermore, he showed that CM satisfies the valuative criterion for properness, and thus CM is a proper algebraic space.

Studying [Hø05], we had trouble following several of the arguments as they appeared to be incomplete or essential conditions seemed not to be satisfied. Here, we construct a scheme W with a representable, surjective and smooth cover $\pi: W \rightarrow CM$. Thereby, we re-prove that CM is an algebraic space. Parts of the construction are based on the ideas in [Hø05]. However, we were able to simplify several of the arguments. Moreover, we give a different, more explicit proof for properness using ideas as in [Kol09].

A similar moduli space was constructed by Alexeev and Knutson. They showed in [AK10] that the space of *branchvarieties* parameterizing reduced curves with finite maps to \mathbb{P}^n is a proper Artin stack. Related moduli spaces can also be found in [Ryd08], [Kol09] and [PT09].

Notation and conventions. All schemes considered here are locally Noetherian. In particular, **Sch** and **Sch_S** denote the categories of locally Noetherian schemes and S -schemes respectively.

Let X and Y be S -schemes, and let $f: X \rightarrow Y$ be a morphism over S . For a base change $g: T \rightarrow S$, we write X_T for the fiber product $X \times_S T$ and f_T for the induced morphism $f \times \text{id}_T: X_T \rightarrow Y_T$ of T -schemes. If $T = \text{Spec}(A)$ for a ring A , we write X_A and f_A instead. Moreover, for $T = \text{Spec}(\kappa(s))$ with $s \in S$, we use the notation X_s and f_s .

Let \mathcal{F} be a quasi-coherent sheaf on a S -scheme X . For a base change $g: T \rightarrow S$, we denote the pullback $h^*\mathcal{F}$, where $h: X_T \rightarrow X$ is the projection, by \mathcal{F}_T . In particular, for $T = \text{Spec}(\kappa(s))$ we write \mathcal{F}_s . This should not be confused with the stalk \mathcal{F}_x of \mathcal{F} at some point $x \in X$.

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2. FAMILIES OF COHEN–MACAULAY CURVES

In this section, we give a definition of the space CM of Cohen–Macaulay curves and investigate some simple examples. Then we show that a Cohen–Macaulay curve does not have nontrivial automorphisms.

Finally, this result is used to prove that CM is a sheaf in the étale topology.

2.1. Definition of the Cohen–Macaulay functor.

Definition 2.1. Let $p(t) = at + b \in \mathbb{Z}[t]$ be a polynomial of degree 1. For a scheme S , let $CM(S) := CM_{\mathbb{P}^n}^{p(t)}(S)$ be the set of isomorphism classes of pairs (C, i) of a flat S -scheme C and a finite S -morphism $i: C \rightarrow \mathbb{P}_S^n$ such that for every $s \in S$ the following properties hold.

- (i) The fiber C_s is a Cohen–Macaulay scheme, that is, all local rings are Cohen–Macaulay rings, and it has pure dimension 1.
- (ii) The induced morphism $i_s: C_s \rightarrow \mathbb{P}_{\kappa(s)}^n$ is an isomorphism onto its image away from a finite set of closed points.
- (iii) The coherent sheaf $(i_*\mathcal{O}_C)_s$ on $\mathbb{P}_{\kappa(s)}^n$ has Hilbert polynomial $p(t)$.

Two such pairs (C_1, i_1) and (C_2, i_2) in $CM(S)$ are considered equal if there exists an S -isomorphism $\alpha: C_1 \xrightarrow{\sim} C_2$ such that the diagram

$$\begin{array}{ccc} & \mathbb{P}_S^n & \\ i_1 \nearrow & & \nwarrow i_2 \\ C_1 & \xrightarrow{\alpha} & C_2 \end{array}$$

commutes.

Remark 2.2. Note that $(i_*\mathcal{O}_C)_s = (i_s)_*\mathcal{O}_{C_s}$ since the morphism i is affine. Moreover, we have $((i_s)_*\mathcal{O}_{C_s})(d) = (i_s)_*(i_s^*\mathcal{O}_{\mathbb{P}_{\kappa(s)}^n}(d))$ for every $d \in \mathbb{Z}$ by the projection formula. Hence it follows for all $r \geq 0$ that $H^r(\mathbb{P}_{\kappa(s)}^n, (i_*\mathcal{O}_C)_s(d)) = H^r(C_s, i_s^*\mathcal{O}_{\mathbb{P}_{\kappa(s)}^n}(d))$. In particular, property (iii) in the definition is equivalent to requiring that the structure sheaf \mathcal{O}_{C_s} has Hilbert polynomial $p(t)$ with respect to the ample invertible sheaf $i_s^*\mathcal{O}_{\mathbb{P}_{\kappa(s)}^n}(1)$.

2.1.1. *Functoriality of CM .* We show in Proposition 2.5 that the assignment CM indeed defines a functor.

Lemma 2.3. *Let $f: X \rightarrow Y$ be a finite morphism of locally Noetherian S -schemes, and let $Z := \text{Supp}(\text{coker}(\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X))$. For $s \in S$, the induced map $f_s: X_s \rightarrow Y_s$ is an isomorphism onto its image away from a finite set of closed points if and only if $\dim(Z_s) = 0$.*

Proof. Let $s \in S$. The finite morphism f_s is an isomorphism onto its image away from the closed locus $\text{Supp}(\text{coker}(f_s^\#))$ in Y_s , where $f_s^\#: \mathcal{O}_{Y_s} \rightarrow (f_s)_*\mathcal{O}_{X_s}$ is the natural map. Now the statement follows as $\text{Supp}(\text{coker}(f_s^\#)) = Z_s$ as closed subsets of Y_s . \square

Lemma 2.4. *Let X be a scheme locally of finite type over a field k , and let $k \subseteq K$ be a field extension.*

- (i) *The scheme X is Cohen–Macaulay if and only if X_K is Cohen–Macaulay.*
- (ii) *The scheme X has pure dimension n if and only if X_K has pure dimension n .*
- (iii) *We have $\dim(X_K) = \dim(X)$.*

Proof. (i) Since that property is local on X , we can reduce to the affine case that is shown in [BH93, Theorem 2.1.10].
(ii) By [Gro65, Corollaire (4.2.8)], the set of dimensions of the irreducible components of X equals the set of dimensions of the irreducible components of X_K , and the statement follows.
(iii) This is [Gro65, Corollaire (4.1.4)]. \square

Proposition 2.5. *The assignment $S \mapsto CM(S)$ defines a contravariant functor $CM: \mathbf{Sch}^\circ \rightarrow \mathbf{Sets}$.*

Proof. Let $f: T \rightarrow S$ be a morphism, and let $(C, i) \in CM(S)$. We claim that $(C_T, i_T) \in CM(T)$. First we observe that being flat and being finite are stable under base change. Moreover, the fiber of C_T over some $t \in T$ is the base change of the fiber of C over the point $f(t)$ by the field extension $\kappa(f(t)) \hookrightarrow \kappa(t)$. As the fiber $C_{f(t)}$ is locally of finite type over $\text{Spec}(\kappa(t))$, it follows from Lemma 2.4 and Lemma 2.3 that $(C_T, i_T) \in CM(T)$. \square

2.1.2. *Cohen–Macaulay rings of dimension 1.* In the case of one-dimensional rings, the Cohen–Macaulay property is particularly simple.

Lemma 2.6. *Let A be a Noetherian ring of dimension 1. The following are equivalent.*

- (i) *The ring A is Cohen–Macaulay.*
- (ii) *All associated prime ideals of A are minimal, that is, A does not have embedded prime ideals.*
- (iii) *Every zero divisor of A is contained in a minimal prime ideal of A .*

Proof. According to the *Unmixedness Theorem* [BH93, Theorem 2.1.6], a ring is Cohen–Macaulay if and only if every ideal I generated by $\text{ht}(I)$ elements is unmixed, that is, all associated prime ideals of I are minimal over I .

As every ideal of maximal height is unmixed and $\dim(A) = 1$, it follows that A is Cohen–Macaulay if and only if the ideal (0) is unmixed. But the latter means that A does not have any embedded prime ideals. This shows the equivalence of assertions (i) and (ii).

That assertion (ii) implies assertion (iii) follows immediately from the fact that the set $\text{Zdv}(A)$ of zero divisors of A equals the union of the associated prime ideals of A . Suppose conversely that every zero divisor is contained in a minimal prime ideal. Then we have the

inclusion

$$\bigcup_{\text{ass. primes } \mathfrak{p}} \mathfrak{p} = \text{Zdv}(A) \subseteq \bigcup_{\text{minimal primes } \mathfrak{p}} \mathfrak{p}.$$

By *Prime avoidance* [Eis95, Lemma 3.3], it follows that every associated prime ideal is minimal, and we are done. \square

2.1.3. Families over a local Artin ring.

Lemma 2.7. *Let X be a scheme of finite type over $\text{Spec}(k)$, where k is a field. Then the following properties are equivalent.*

- (i) *The scheme X has pure dimension n .*
- (ii) *Every nonempty open subscheme U of X has pure dimension n .*

Proof. First we can without loss of generality assume that X and hence also every open subscheme of X is reduced. Moreover, we observe that $\dim(V) = \dim(Y)$ for every nonempty open subscheme V of an integral scheme Y of finite type over k , see [Gro65, (4.1.1.3)].

Suppose that X has pure dimension n , and let $\emptyset \neq U \subseteq X$ be an open subscheme. The irreducible components of U are of the form $Y \cap U$ where Y is an irreducible component of X . By the remark above, it follows that $\dim(Y \cap U) = \dim(Y) = n$.

The opposite implication follows directly by setting $U = X$. \square

Lemma 2.8. *Let X be a scheme of finite type over $\text{Spec}(R)$, where R is a local Artin ring. Suppose that the closed fiber X_0 has pure dimension n . Then X has pure dimension n . Moreover, every nonempty open subscheme U of X has pure dimension n .*

Proof. Observe first that every nonempty open subscheme V of X_0 has pure dimension n by Lemma 2.7. Now the statement follows immediately since X and X_0 have the same underlying topological space. \square

Proposition 2.9. *Let $f: X \rightarrow S$ be a flat morphism of locally Noetherian schemes.*

- (i) *Suppose that S and all fibers X_s , where $s \in S$, are Cohen–Macaulay schemes. Then X is Cohen–Macaulay.*
- (ii) *Let $S = \text{Spec}(R)$ for a local Artin ring R , and suppose that the closed fiber X_0 is Cohen–Macaulay and of pure dimension n . Then X is Cohen–Macaulay and of pure dimension n .*

Proof. Since the statement is local on both X and S , we can without loss of generality assume that $X = \text{Spec}(A)$ and $S = \text{Spec}(R)$ are affine, where A is a flat R -algebra and R is Cohen–Macaulay. Then assertion (i) is the statement of [BH93, Proposition 2.1.16(b)].

For assertion (ii), we observe first that S is Cohen–Macaulay. In particular, the scheme X is Cohen–Macaulay by assertion (i). Moreover, it follows from Lemma 2.8 that X has pure dimension n . \square

Proposition 2.10. *Let $(C, i) \in CM(\text{Spec}(R))$ for a local Artin ring R . Then C is Cohen–Macaulay and of pure dimension 1. Moreover, the morphism i is an isomorphism onto its image away from finitely many closed points.*

Proof. We have seen in Proposition 2.9 that the scheme C is Cohen–Macaulay and of pure dimension 1.

Let $Y = \text{Supp}(\text{coker}(\mathcal{O}_{\mathbb{P}_R^n} \rightarrow i_*\mathcal{O}_C))$. Then the closed fiber Y_0 is zero-dimensional by Lemma 2.3. Now the statement follows as Y and Y_0 have the same underlying topological space and, in particular, the same dimension. \square

2.1.4. Scheme-theoretic image. Next we study the scheme-theoretic image $i(C)$ of a point $(C, i) \in CM(\text{Spec}(k))$. In particular, we show that it is Cohen–Macaulay and of pure dimension 1.

Lemma 2.11. *Let $j: B \hookrightarrow A$ be a finite injective homomorphism of Noetherian rings, and suppose that A is Cohen–Macaulay and of pure dimension 1. Then B is Cohen–Macaulay of pure dimension 1.*

Proof. By *Cohen–Seidenberg Theorem* [Mat80, (5.E) Theorem 5], we have $\dim(B) = \dim(A)$ and every minimal prime \mathfrak{p} of B is contained in a minimal prime \mathfrak{q} of A with $\dim(B/\mathfrak{p}) = \dim(A/\mathfrak{q})$. It follows that B has pure dimension 1.

By Lemma 2.6, a one-dimensional ring is Cohen–Macaulay if and only if every zero divisor is contained in a minimal prime ideal. So let b be a zero divisor of B . Then its image $j(b)$ is a zero divisor of A , and hence it is contained in a minimal prime ideal \mathfrak{q} of A . The restriction $\mathfrak{p} := j^{-1}(\mathfrak{q})$ is a prime ideal of B containing b with $\dim(B/\mathfrak{p}) = \dim(A/\mathfrak{q})$. It follows that $\dim(B/\mathfrak{p}) = 1$, and \mathfrak{p} is minimal. \square

Proposition 2.12. *Let $f: X \rightarrow Y$ be a finite morphism of schemes of finite type over a field. Suppose that X is Cohen–Macaulay and of pure dimension 1. Then the scheme-theoretic image Z of X in Y is Cohen–Macaulay and of pure dimension 1.*

Proof. By definition and Lemma 2.7 respectively, both conditions are local, and we reduce to the case that $X = \text{Spec}(A)$ and $Y = \text{Spec}(C)$ are affine. Then $Z = \text{Spec}(B)$ with $B = C/\ker(C \rightarrow A)$. By Lemma 2.11, the ring B is Cohen–Macaulay of pure dimension 1. \square

Corollary 2.13. *Let $(C, i) \in CM(\text{Spec}(k))$ for a field k . Then the scheme-theoretic image of C in \mathbb{P}_k^n is Cohen–Macaulay and of pure dimension 1.*

Proof. This follows directly from Proposition 2.12. \square

Moreover, we can give bounds for the degree and the genus of the projective curve $i(C)$.

Proposition 2.14. *Let k be a field, and let $(C, i) \in CM_{\mathbb{P}^n}^{at+b}(\text{Spec}(k))$. Then the scheme-theoretic image $i(C)$ of C in \mathbb{P}_k^n is a curve of degree a and arithmetic genus g that is bounded by*

$$1 - b \leq g \leq \frac{1}{2}(a - 1)(a - 2).$$

In particular, there are only finitely many possibilities for the Hilbert polynomial $p_{i(C)}(t)$ of the curve $i(C)$.

Proof. The finite morphism i factors through the scheme-theoretic image as

$$\begin{array}{ccc} C & \xrightarrow{i} & \mathbb{P}_k^n \\ & \searrow h \quad \nearrow j & \\ & i(C) & \end{array}$$

where h is finite and an isomorphism away from finitely many closed points, and j is a closed immersion. The one-dimensional scheme $i(C)$ has Hilbert polynomial of degree 1, say $p_{i(C)}(t) = ct + d$.

The cokernel \mathcal{K} of the injective map $j_*\mathcal{O}_{i(C)} \rightarrow i_*\mathcal{O}_C$ is supported on the finitely many closed points that form the non-isomorphism locus of h . Hence its Hilbert polynomial $p_{\mathcal{K}}(t)$ is constant, $p_{\mathcal{K}}(t) = h^0(\mathcal{K}) =: l$ for $l \in \mathbb{N}$. Consider the short exact sequence

$$0 \longrightarrow j_*\mathcal{O}_{i(C)} \longrightarrow i_*\mathcal{O}_C \longrightarrow \mathcal{K} \longrightarrow 0$$

of $\mathcal{O}_{\mathbb{P}_k^n}$ -modules. The additivity of the Hilbert polynomial implies then that

$$p_{i_*\mathcal{O}_C}(t) = p_{i(C)}(t) + p_{\mathcal{K}}(t), \tag{1}$$

that is, $at + b = ct + d + l$. In particular, we get that $c = a$ and $d \leq b = d + l$. The relation $d = 1 - g$ gives the lower bound $1 - b \leq g$. Moreover, the arithmetic genus of a curve of degree a is bounded from above by $\frac{1}{2}(a - 1)(a - 2)$ by [Har94, Theorem 3.1]. Note that the proof there works even for curves in \mathbb{P}_k^n with $n \geq 3$. \square

2.2. First examples. The results of Proposition 2.14 allow us to understand some simple examples.

Theorem 2.15. *Let $p(t) = at + b \in \mathbb{Z}[t]$ be a polynomial.*

- (i) *Suppose that $b < 1 - \frac{1}{2}(a - 1)(a - 2)$. Then $CM_{\mathbb{P}^n}^{at+b} = \emptyset$.*
- (ii) *If $b = 1 - \frac{1}{2}(a - 1)(a - 2)$, then $CM_{\mathbb{P}^n}^{at+b} = \text{Hilb}_{\mathbb{P}^n}^{at+b}$ is the Hilbert scheme of plane curves of degree a in \mathbb{P}^n .*

Proof. Let k be a field, and let $(C, i) \in CM(\text{Spec}(k))$. Then by Proposition 2.14, the scheme-theoretic image $i(C)$ of C in \mathbb{P}_k^n is a curve of degree a and arithmetic genus g such that $1 - b \leq g \leq 1 - \frac{1}{2}(a - 1)(a - 2)$. This shows directly that no such curves exist if $b < 1 - \frac{1}{2}(a - 1)(a - 2)$, and hence assertion (i).

If $b = 1 - \frac{1}{2}(a-1)(a-2)$, it follows that $g = 1 - b$. Observe that the image $i(C)$ has Hilbert polynomial $p(t) = p_{i_*\mathcal{O}_C}(t)$ in this case. Then Equation (1) above implies that $\text{coker}(\mathcal{O}_{\mathbb{P}_k^n} \rightarrow i_*\mathcal{O}_C) = 0$, that is, the curves C and $i(C)$ are isomorphic, and the map i is a closed immersion. Since being a closed immersion can be checked on the fibers, see [Gro61b, Proposition (4.6.7)], it follows that for every scheme S and every family $(C, i) \in CM(S)$ the map i is a closed immersion. Conversely, every closed subscheme of \mathbb{P}_k^n with Hilbert polynomial $p(t) = at + 1 - \frac{1}{2}(a-1)(a-2)$ is a plane curve without embedded or isolated points by [Har94, Theorem 3.1]. \square

2.3. Automorphisms of CM curves. Next we show that for every scheme S , and every $(C, i) \in CM(S)$, the scheme C does not have any nontrivial automorphism over \mathbb{P}_S^n .

Lemma 2.16. *Let $\varphi: B \rightarrow A$ be a finite homomorphism of Noetherian rings, where A is Cohen–Macaulay and of pure dimension 1. Suppose that the localized map $\varphi_{\mathfrak{p}}: B_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}}$ is surjective for all prime ideals \mathfrak{p} of B apart from finitely many maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_s$. Then there exists an element $b \in B$ that is not a zero divisor of A such that the map $\varphi_b: B_b \rightarrow A_b$ is surjective.*

Proof. An element b with the required properties has to be contained in all exceptions $\mathfrak{m}_1, \dots, \mathfrak{m}_s$. Moreover, in order not to be a zero divisor of A , the element b may not be contained in any of the associated prime ideals of A as a B -module. We claim that such an element exists, that is, $\bigcap_{i=1}^s \mathfrak{m}_i \not\subseteq \bigcup_{j=1}^t \mathfrak{p}_j$, where we set $\text{Ass}_B(A) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$. Suppose namely the opposite, that is, $\bigcap_{i=1}^s \mathfrak{m}_i \subseteq \bigcup_{j=1}^t \mathfrak{p}_j$. Prime avoidance implies then that $\mathfrak{m}_i \subseteq \mathfrak{p}_j$ for some i and j . In particular, it follows that \mathfrak{p}_j is a maximal ideal of B . Note that by [Mat80, (9.A) Proposition], we have $\text{Ass}_B(A) = \{\varphi^{-1}(\mathfrak{q}) \mid \mathfrak{q} \in \text{Ass}_A(A)\}$, and hence $\mathfrak{p}_j = \varphi^{-1}(\mathfrak{q})$ for some $\mathfrak{q} \in \text{Ass}_A(A)$. The induced map $B/\mathfrak{p}_j \hookrightarrow A/\mathfrak{q}$ is a finite and hence integral extension of a field, and therefore A/\mathfrak{q} is a field. This contradicts the fact that \mathfrak{q} is an associated prime ideal of the Cohen–Macaulay ring A of pure dimension 1. \square

Lemma 2.17. *Let R be a Noetherian ring, and let M be a finitely generated R -module. Then the natural map $M \rightarrow \prod_{\mathfrak{p} \in \text{Ass}(M)} M_{\mathfrak{p}}$ is injective.*

Proof. Suppose that the element $m \in M$ is mapped to 0. Then for every associated prime ideal $\mathfrak{p} \in \text{Ass}(M)$ there exists $s \notin \mathfrak{p}$ such that $sm = 0$. In particular, $\text{Ann}(m) \not\subseteq \mathfrak{p}$ for all associated prime ideals \mathfrak{p} of M . It follows that $\text{Ann}(m) = R$, and hence $m = 0$. \square

Lemma 2.18. *Let R be a Noetherian ring. Then R has an embedding into a finite product of local Artin rings.*

Proof. Let $(0) = \bigcap_{j=1}^l \mathfrak{q}_j$ be a *primary decomposition* of the zero ideal of R , see [Bou72, Theorem IV.2.2.1]. The natural map $R \rightarrow \prod_{j=1}^l R/\mathfrak{q}_j$ is then injective. The ideal \mathfrak{q}_j is \mathfrak{p}_j -primary for $\mathfrak{p}_j := \sqrt{\mathfrak{q}_j}$, and hence we have that $\text{Ass}(R/\mathfrak{q}_j) = \{\mathfrak{p}_j/\mathfrak{q}_j\}$. In particular, it follows by Lemma 2.17 that the localization map $R/\mathfrak{q}_j \rightarrow R_j := (R/\mathfrak{q}_j)_{\mathfrak{p}_j/\mathfrak{q}_j}$ is injective. Since \mathfrak{p}_j is a minimal prime over ideal of \mathfrak{q}_j , the ring R_j is a local Artin ring, and the composition $R \hookrightarrow \prod_{j=1}^l R/\mathfrak{q}_j \hookrightarrow \prod_{j=1}^l R_j$ has the required properties. \square

Theorem 2.19. *Let $(C, i) \in CM(S)$ for a locally Noetherian scheme S . Let α be an S -automorphism of C that is compatible with the finite morphism i , that is, $i \circ \alpha = i$. Then α is the identity morphism on C .*

Proof. Suppose first that $S = \text{Spec}(R)$ for a local Artin ring R . By Proposition 2.10, the scheme C is Cohen–Macaulay and of pure dimension 1, and i is an isomorphism onto its image away from finitely many closed points. Let $V = \text{Spec}(B) \subset \mathbb{P}_R^n$ be an open subset. Then $i^{-1}(V) = \text{Spec}(A)$ is affine, and α induces an endomorphism of $\text{Spec}(A)$. In particular, we can reduce to the affine situation, and we have to show that every B -algebra endomorphism α of A is the identity. Note that the ring A is Cohen–Macaulay and of pure dimension 1, and the structure map $\varphi: B \rightarrow A$ satisfies the conditions of Lemma 2.16. Hence there exists $b \in B$ such that $B_b \rightarrow A_b$ is surjective and b is not a zero divisor of A . It follows that the induced B_b -algebra endomorphism α_b is the identity. Moreover, we have an inclusion $A \subseteq A_b$. Now the statement follows as α is the restriction of α_b to A .

Secondly, we consider the case that $S = \text{Spec}(R)$ for a Noetherian ring R . As in the first case it suffices to show that for any open affine subset $V = \text{Spec}(B) \subset \mathbb{P}_R^n$ and $i^{-1}(V) = \text{Spec}(A)$, every B -algebra endomorphism α of A is the identity. Let $R \hookrightarrow \prod_{j=1}^l R_j$ be an embedding into a finite product of local Artin rings R_j , see Lemma 2.18. Since A is flat over R , we get an injection $A \hookrightarrow A \otimes_R \prod_{j=1}^l R_j \cong \prod_{j=1}^l (A \otimes_R R_j)$. Moreover, α is the restriction of the endomorphism $\prod_j (\alpha \otimes \text{id}_{R_j})$ of $\prod_j (A \otimes_R R_j)$ to A . But $\alpha \otimes \text{id}_{R_j}$ is a $B \otimes_R R_j$ -algebra endomorphism of $A \otimes_R R_j$, and hence, as in the first case, it is the identity. It follows that $\alpha = \text{id}$.

For a general locally Noetherian scheme S , we observe that the statement is local on S , and we can directly reduce to the affine case. \square

Corollary 2.20. *Let S be a scheme, and let $(C_1, i_1), (C_2, i_2) \in CM(S)$. There exists at most one isomorphism $\beta: C_1 \rightarrow C_2$ such that $i_2 \circ \beta = i_1$.*

Proof. Let β_1, β_2 be two such isomorphisms. Then $\alpha := \beta_2^{-1} \circ \beta_1$ is an automorphism of C_1 such that $i_1 \circ \alpha = i_1$, and hence $\beta_1 = \beta_2$ by Theorem 2.19. \square

Remark 2.21. Note that Theorem 2.19 does not hold if we drop the requirement in Definition 2.1 of CM that the curves C_s have to be of pure dimension 1. Consider namely the k -scheme C that is the disjoint union of a line and two isolated points x_1 and x_2 , and a finite morphism $i: C \rightarrow \mathbb{P}_k^n$ that embeds the line and maps the points x_1 and x_2 to the same image. Then the morphism $\alpha: C \rightarrow C$ that keeps the line fixed and exchanges the points x_1 and x_2 is a nontrivial automorphism such that $i \circ \alpha = i$.

2.4. Sheaf in the étale topology. The nonexistence of nontrivial automorphisms is an important tool for showing that CM is a sheaf in the étale topology.

Proposition 2.22 ([Mil80, Proposition II.1.5]). *Let $F: \mathbf{Sch}^\circ \rightarrow \mathbf{Sets}$ be a functor. Then F is a sheaf in the étale topology if and only if the following properties hold.*

- (i) *F is a sheaf in the Zariski topology.*
- (ii) *For any surjective étale morphism $V \rightarrow U$ of affine schemes the diagram*

$$F(U) \rightarrow F(V) \rightrightarrows F(V \times_U V)$$

is exact.

Proposition 2.23. *The functor CM is a sheaf in the Zariski topology.*

Proof. Let S be a scheme, and let $\{S_a\}$ be an open cover of S . We have to show that the sequence

$$CM(S) \rightarrow \prod_a CM(S_a) \rightrightarrows \prod_{a,b} CM(S_{ab}),$$

where we write S_{ab} for the intersection $S_a \cap S_b$, is exact.

Let first (C, i) and (D, j) be two elements of $CM(S)$ restricting to the same element of $CM(S_a)$ for every a . Then for every a there exists an isomorphism $\alpha_a: C_{S_a} \xrightarrow{\sim} D_{S_a}$ that commutes with i and j . Restriction of α_a and α_b to the intersection $C_{S_{ab}}$ gives an automorphism $\alpha_b^{-1} \circ \alpha_a$ of $C_{S_{ab}}$ that commutes with the finite morphism i . Since $\alpha_b^{-1} \circ \alpha_a$ is the identity morphism by Theorem 2.19, we see that the morphisms α_a coincide on the overlaps and hence glue to an isomorphism $\alpha: C \xrightarrow{\sim} D$ with $j \circ \alpha = i$. This shows that $(C, i) = (D, j)$ as elements of $CM(S)$, and the first map is injective.

Now let $\{(C_a, i_a) \in CM(S_a)\}$ be a family that has the same image in $\prod_{a,b} CM(S_{ab})$ under both restriction maps. That means that there exist isomorphisms $\alpha_{ab}: (C_a)_{S_{ab}} \xrightarrow{\sim} (C_b)_{S_{ab}}$ that commute with the restrictions of i_a and i_b . By Theorem 2.19 and Corollary 2.20, we have equalities $\alpha_{ba} = \alpha_{ab}^{-1}$ and $\alpha_{ac} = \alpha_{bc} \circ \alpha_{ab}$. Hence the $\{C_a\}$ glue to a scheme C over S with a morphism $i: C \rightarrow \mathbb{P}_S^n$ such that $i|_{C_a} = i_a$ for every a . As the defining properties of CM are local on S , we see that $(C, i) \in CM(S)$. \square

Let $p: S' \rightarrow S$ be a morphism of schemes. We have projection morphisms $p_i: S' \times_S S' \rightarrow S'$ for $i = 1, 2$ and $p_{ij}: S' \times_S S' \times_S S' \rightarrow S' \times_S S'$ for $1 \leq i < j \leq 3$. A *descent datum* is a scheme X' over S' together with an $(S' \times_S S')$ -isomorphism $\varphi: p_1^* X' \xrightarrow{\sim} p_2^* X'$ such that $p_{23}^* \varphi \circ p_{12}^* \varphi = p_{13}^* \varphi$. In particular, for a S -scheme X the pullback $p^* X = X_{S'}$ together with the natural isomorphism $\varphi_X: p_1^* p^* X \cong (p \circ p_1)^* X = (p \circ p_2)^* X \cong p_2^* p^* X$ defines the so-called canonical descent datum.

Let $\mathbf{Sch}_{S'/S}$ be the category where the objects are descent data, and morphisms between descent data (X'_1, φ_1) and (X'_2, φ_2) are S' -morphisms $\tau: X'_1 \rightarrow X'_2$ such that $\varphi_2 \circ p_1^* \tau = p_2^* \tau \circ \varphi_1$. Then there is a functor $G: \mathbf{Sch}_S \rightarrow \mathbf{Sch}_{S'/S}$ that maps a S -scheme X to the canonical descent datum $(p^* X, \varphi_X)$.

Theorem 2.24 ([GW10, Theorem 14.70]). *Let $p: S' \rightarrow S$ be a faithfully flat and quasi-compact morphism of schemes.*

- (i) *The functor $G: \mathbf{Sch}_S \rightarrow \mathbf{Sch}_{S'/S}$, $X \mapsto (p^* X, \varphi_X)$ is fully faithful.*
- (ii) *Let (X', φ) be a descent datum, and suppose that X' is affine over S' . Then there exists a scheme X that is affine over S and such that $(X', \varphi) \cong (p^* X, \varphi_X)$.*

Proposition 2.25. *Let $V \rightarrow U$ be a faithfully flat morphism of affine schemes U and V . Then the sequence*

$$CM(U) \rightarrow CM(V) \rightrightarrows CM(V \times_U V)$$

is exact.

Proof. Being faithfully flat and quasi-compact is stable under base change. Hence also the induced map $\mathbb{P}_V^n \rightarrow \mathbb{P}_U^n$ is faithfully flat and quasi-compact. We will use results on descent as in Theorem 2.24 with $S' = \mathbb{P}_V^n$ and $S = \mathbb{P}_U^n$.

Let (C, i) and (D, j) be two elements of $CM(U)$ having the same image in $CM(V)$. Then there exists an isomorphism $\alpha: C_V \xrightarrow{\sim} D_V$ with $i_V = j_V \circ \alpha$. By Theorem 2.19, we have that $\varphi_{D_V} \circ p_1^* \alpha = p_2^* \alpha \circ \varphi_{C_V}$, and hence α is an isomorphism between the descent data (C_V, φ_{C_V}) and (D_V, φ_{D_V}) in $\mathbf{Sch}_{\mathbb{P}_V^n / \mathbb{P}_U^n}$. Since the functor $G: \mathbf{Sch}_{\mathbb{P}_U^n} \rightarrow \mathbf{Sch}_{\mathbb{P}_V^n / \mathbb{P}_U^n}$ is fully faithful by Theorem 2.24, it follows that C and D are isomorphic as schemes over \mathbb{P}_U^n , and hence $(C, i) = (D, j)$ in $CM(U)$.

Now suppose that an element $(C_V, i_V) \in CM(V)$ has the same image in $CM(V \times_U V)$ under both projections. Then we have an isomorphism $\alpha: p_1^* C_V \xrightarrow{\sim} p_2^* C_V$ of schemes over $\mathbb{P}_{V \times_U V}^n$. Again by Theorem 2.19, α satisfies the cocycle condition so we get a descent datum (C_V, α) . Since C_V is affine over $S' = \mathbb{P}_V^n$, there exists by Theorem 2.24 a scheme C with an affine morphism $i: C \rightarrow \mathbb{P}_U^n$ such that $C \times_U V \cong C_V$ as schemes over \mathbb{P}_V^n . We claim that $(C, i) \in CM(U)$. As the properties finite and flat are stable under faithfully flat descent by [FGI⁺05, Proposition 1.2.36], the morphism i is finite and C is flat over U . Note further that the

map $V \rightarrow U$ is surjective. Hence every fiber of C over U gives a fiber of C_V over V after a suitable change of base field. Then it follows from Lemma 2.4 that (C, i) indeed is an element of $CM(U)$. \square

Theorem 2.26. *The functor CM is a sheaf in the étale topology.*

Proof. We have seen in Proposition 2.23 that CM is a sheaf in the Zariski topology. Moreover, since a surjective and étale morphism is faithfully flat, the sheaf property is satisfied for affine étale coverings by Proposition 2.25. Now the statement follows from Proposition 2.22. \square

3. REPRESENTABILITY OF THE DIAGONAL

In this section, we show that the diagonal map $CM \rightarrow CM \times CM$ is representable. In particular, this implies that every map $X \rightarrow CM$, where X is a scheme, is representable.

Proposition 3.1. *Let S be a locally Noetherian scheme, and let X and Y be schemes that are projective over S . Suppose further that X is flat over S . The functors $(\mathbf{Sch}_S)^\circ \rightarrow \mathbf{Sets}$ defined by*

$$\begin{aligned} \mathcal{H}om_S(X, Y)(T) &:= \mathrm{Hom}_T(X_T, Y_T), \\ \mathcal{I}som_S(X, Y)(T) &:= \mathrm{Isom}_T(X_T, Y_T), \end{aligned}$$

for every S -scheme T , are represented by schemes locally of finite type over S .

Proof. Since the statement is local on S by [GW10, Theorem 8.9], we can assume that S is Noetherian, and the proposition follows from [Gro61c, Section 4.c, p.221-19f.] \square

Corollary 3.2. *Let S be locally Noetherian, and let $f_1: X_1 \rightarrow Y$ and $f_2: X_2 \rightarrow Y$ be morphisms of projective S -schemes. Suppose that X_1 and X_2 are flat over S . Let*

$$\mathcal{I} := \mathcal{I}som_S((X_1, f_1), (X_2, f_2)): (\mathbf{Sch}_S)^\circ \rightarrow \mathbf{Sets}$$

be the functor defined by

$$\mathcal{I}(T) = \{\alpha \in \mathcal{I}som_S(X_1, X_2)(T) \mid f_{1,T} = f_{2,T} \circ \alpha\}$$

for every S -scheme T . Then \mathcal{I} is represented by a scheme locally of finite type over S .

Proof. Let $\Delta: \mathcal{H}om_S(X_1, Y) \rightarrow \mathcal{H}om_S(X_1, Y) \times \mathcal{H}om_S(X_1, Y)$ be the diagonal map. Consider further the map

$$\tau: \mathcal{I}som_S(X_1, X_2) \rightarrow \mathcal{H}om_S(X_1, Y) \times \mathcal{H}om_S(X_1, Y)$$

given by $\tau(T)(\alpha) = (f_{1,T}, f_{2,T} \circ \alpha)$. Then the diagram

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{\quad} & \mathcal{I}som_S(X_1, X_2) \\ \downarrow & & \downarrow \tau \\ \mathcal{H}om_S(X_1, Y) & \xrightarrow{\Delta} & \mathcal{H}om_S(X_1, Y) \times \mathcal{H}om_S(X_1, Y) \end{array}$$

is Cartesian. Since the functors $\mathcal{I}som_S(X_1, X_2)$ and $\mathcal{H}om_S(X_1, Y)$ are representable by Proposition 3.1, it follows that the fiber product \mathcal{I} is representable. Note moreover that Δ and hence $\mathcal{I} \rightarrow \mathcal{I}som_S(X_1, X_2)$ is locally of finite type. As $\mathcal{I}som_S(X_1, X_2)$ is locally of finite type over S , it follows now that \mathcal{I} is locally of finite type over S . \square

Lemma 3.3. *Let S be a locally Noetherian scheme, and let (C_1, i_1) and (C_2, i_2) be elements of $CM(S)$. Let $\mathcal{I} = \mathcal{I}som_S((C_1, i_1), (C_2, i_2))$ be the functor defined in Corollary 3.2. Then \mathcal{I} is represented by a scheme Y such that the structure morphism $Y \rightarrow S$ is a monomorphism locally of finite type.*

Proof. Let Y be the S -scheme that represents \mathcal{I} by Corollary 3.2, and let $h: Y \rightarrow S$ be the structure morphism. Let $g_1, g_2: T \rightarrow Y$ be two morphisms from a scheme T such that $h \circ g_1 = h \circ g_2$. Taking this composition as structure map over S , we can consider g_1 and g_2 as elements of $\text{Hom}_S(T, Y) = \mathcal{I}(T)$. So g_1 and g_2 correspond to two isomorphisms $\alpha_1, \alpha_2: C_{1,T} \xrightarrow{\sim} C_{2,T}$ over \mathbb{P}_T^n . Since every set $\mathcal{I}(T)$ contains at most one element by Corollary 2.20, it follows that $\alpha_1 = \alpha_2$. But this means $g_1 = g_2$, and we get that h is a monomorphism. \square

Proposition 3.4. *The diagonal $CM \rightarrow CM \times CM$ is representable and locally of finite type.*

Proof. Let S be a locally Noetherian scheme, and $S \rightarrow CM \times CM$ a map corresponding to elements $(C_1, i_1), (C_2, i_2) \in CM(S)$. We have to show that the fiber product $CM \times_{CM \times CM} S$ is representable and that the second projection $CM \times_{CM \times CM} S \rightarrow S$ is locally of finite type.

For every morphism $g: T \rightarrow S$, the composition $T \xrightarrow{g} S \rightarrow CM \times CM$ is represented by the elements $(C_{1,T}, i_{1,T})$ and $(C_{2,T}, i_{2,T})$ of $CM(T)$. It follows that

$$(CM \times_{CM \times CM} S)(T) = \left\{ g: T \rightarrow S \mid \begin{array}{c} (C_{1,T}, i_{1,T}) = (C_{2,T}, i_{2,T}) \\ \text{in } CM(T) \end{array} \right\}.$$

By definition of CM , the two pairs $(C_{1,T}, i_{1,T})$ and $(C_{2,T}, i_{2,T})$ are equal if there exists an isomorphism $\alpha: C_{1,T} \xrightarrow{\sim} C_{2,T}$ over T such that $i_{1,T} = i_{2,T} \circ \alpha$. In other words, we have that

$$(CM \times_{CM \times CM} S)(T) = \{g: T \rightarrow S \mid \mathcal{I}som_S((C_1, i_1), (C_2, i_2))(T) \neq \emptyset\}.$$

Note that the last expression makes sense since we can consider T as an S -scheme via g . By Lemma 3.3, the functor $\mathcal{I}som_S((C_1, i_1), (C_2, i_2))$ is represented by a scheme Y , and the structure morphism $Y \rightarrow S$ is a monomorphism. Note that a morphism $g: T \rightarrow S$ factors through Y if and only if $\text{Hom}_S(T, Y) \neq \emptyset$. As $Y \rightarrow S$ is a monomorphism, such a factorization is unique and hence

$$(CM \times_{CM \times CM} S)(T) = \text{Hom}(T, Y).$$

This shows that the fiber product $CM \times_{CM \times CM} S$ is represented by the scheme Y that is locally of finite type over S . \square

Representability of the diagonal is equivalent to any map from a scheme being representable.

Proposition 3.5. *Let $F: \mathbf{Sch}^\circ \rightarrow \mathbf{Sets}$ be a functor. Then the following properties are equivalent.*

- (i) *The diagonal $F \rightarrow F \times F$ is representable.*
- (ii) *For every scheme U and every $\xi \in F(U)$ the map $\xi: U \rightarrow F$ is representable.*

Proof. Suppose first that the diagonal $F \rightarrow F \times F$ is representable, and let $\xi: U \rightarrow F$ and $\eta: V \rightarrow F$ be two maps from schemes U and V . We have to show that the fiber product $U \times_F V$ is representable. Consider the map $\xi \times \eta: U \times V \rightarrow F \times F$. Then the fiber product $(U \times V) \times_{F \times F} F$ is representable by assumption. As $(U \times V) \times_{F \times F} F \cong U \times_F V$, we see that assertion (i) implies assertion (ii).

Suppose conversely that assertion (ii) holds. Let $(\xi, \xi'): V \rightarrow F \times F$ be a map from a scheme V to the product $F \times F$. By assumption, the fiber product $V \times_F V$ with respect to the maps ξ and ξ' is represented by a scheme W . Moreover, one checks that the diagram

$$\begin{array}{ccc} W \times_{V \times V} V & \longrightarrow & V \\ \downarrow & & \downarrow \\ F & \longrightarrow & F \times F \end{array}$$

is Cartesian. This shows that the diagonal is representable, and hence assertion (i) holds. \square

Corollary 3.6. *Let $\xi: X \rightarrow CM$ be a map, where X is a scheme locally of finite type over $\mathrm{Spec}(\mathbb{Z})$. Then ξ is representable and locally of finite type.*

Proof. For any map $\eta: Y \rightarrow CM$ the diagram

$$\begin{array}{ccc} X \times_{CM} Y & \longrightarrow & X \times Y \\ \downarrow & & \downarrow \xi \times \eta \\ CM & \xrightarrow{\Delta} & CM \times CM \end{array}$$

is Cartesian. Since the diagonal Δ is representable and locally of finite type, we see that $X \times_{CM} Y \rightarrow X \times Y$ is locally of finite type. Now the statement follows as the projection $X \times Y \rightarrow Y$ is locally of finite type by assumption on X . \square

4. EMBEDDING IN PROJECTIVE SPACE

The goal of this section is to show that there exists a positive integer N such that for every base scheme S and any $(C, i) \in CM(S)$ there exists an open cover $\{S_a\}$ of S such that C_{S_a} can be embedded into the projective space $\mathbb{P}_{S_a}^N$.

First we consider the case of curves over fields. We show in Proposition 4.14 that there exist integers m and $N := N(m)$ such that for every field k and every $(C, i) \in CM(\text{Spec}(k))$ the invertible sheaf $\mathcal{L} := i^* \mathcal{O}_{\mathbb{P}_k^n}(m)$ is very ample and every choice of basis of $H^0(C, \mathcal{L})$ gives rise to a closed immersion $j: C \hookrightarrow \mathbb{P}_k^N$.

The main tool here is *Castelnuovo–Mumford regularity* that also plays an important roll in the construction of the Hilbert scheme. In Subsection 4.1, we give an introduction to this concept.

The case of a general base scheme S is then treated in Subsection 4.3.

4.1. Some facts about Castelnuovo–Mumford regularity. In the following, we sometimes suppress the projective space \mathbb{P}_k^n in the notation of the cohomology groups $H^i(\mathbb{P}_k^n, \mathcal{F})$ for a coherent sheaf \mathcal{F} in order to increase readability. Moreover, we write $h^i(\mathcal{F}) := \dim_k H^i(\mathcal{F})$ for its dimension.

Definition 4.1. Let \mathbb{P}_k^n be the projective n -space over a field k , and let \mathcal{F} be a coherent sheaf on \mathbb{P}_k^n . Let $m \in \mathbb{Z}$ be an integer. If

$$H^i(\mathbb{P}_k^n, \mathcal{F}(m - i)) = 0$$

for every $i > 0$, then \mathcal{F} is *m -regular*.

Example 4.2. Suppose that \mathcal{F} is a coherent sheaf on \mathbb{P}_k^n with zero-dimensional support. Then \mathcal{F} is m -regular for every $m \in \mathbb{Z}$ since the cohomology vanishes in degrees that exceed the dimension of the support.

Example 4.3. For every integer d , the invertible sheaf $\mathcal{O}(d) := \mathcal{O}_{\mathbb{P}_k^n}(d)$ on \mathbb{P}_k^n is $(-d)$ -regular. Indeed, we have that $H^i(\mathcal{O}(d + m - i)) = 0$ unless $i = 0, n$ and $H^n(\mathcal{O}(d + m - n)) \cong H^0(\mathcal{O}(-d - m - 1))^\vee$ by [Har77, Theorem III.5.1]. Clearly the last term vanishes for $m \geq -d$.

Proposition 4.4 ([Kle71, Proposition 1.3]). *Let \mathcal{F} be a m -regular coherent sheaf on \mathbb{P}_k^n . Then for every $m' \geq m$ we have the following.*

- (i) \mathcal{F} is m' -regular.
- (ii) $\mathcal{F}(m')$ is generated by its global sections.

Corollary 4.5. *Let \mathcal{F} be m -regular. Then $H^i(\mathbb{P}_k^n, \mathcal{F}(m)) = 0$ for $i > 0$, and hence $h^0(\mathcal{F}(m)) = p_{\mathcal{F}}(m)$ for the Hilbert polynomial $p_{\mathcal{F}}(t)$ of \mathcal{F} .*

Proof. As $H^i(\mathbb{P}_k^n, \mathcal{F}(m)) = H^i(\mathbb{P}_k^n, \mathcal{F}((m + i) - i))$, the statement is a direct consequence of assertion (i) in the proposition. \square

Lemma 4.6. *Let $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ be an exact sequence of coherent sheaves on \mathbb{P}_k^n . Then we have the following.*

- (i) *If \mathcal{F} and \mathcal{H} are m -regular, then \mathcal{G} is m -regular.*
- (ii) *Suppose that \mathcal{G} is m -regular and that $H^i(\mathbb{P}_k^n, \mathcal{F}(m + 1 - i)) = 0$ for all $i > 1$ (in particular, this is satisfied if \mathcal{F} is $(m + 1)$ -regular). Then \mathcal{H} is m -regular.*

Proof. The statements follow directly from the induced long exact sequence

$$\cdots H^i(\mathcal{F}(d)) \longrightarrow H^i(\mathcal{G}(d)) \longrightarrow H^i(\mathcal{H}(d)) \longrightarrow H^{i+1}(\mathcal{F}(d)) \cdots \quad (2)$$

in cohomology for $d = m - i$.

For assertion (i) suppose that \mathcal{F} and \mathcal{H} are m -regular. Then we have that $H^i(\mathcal{F}(m-i)) = H^i(\mathcal{H}(m-i)) = 0$ for all $i > 0$. From sequence (2) we get that $H^i(\mathcal{G}(m-i)) = 0$ for all $i > 0$, and \mathcal{G} is m -regular.

Moreover, if $H^i(\mathcal{G}(m-i)) = H^{i+1}(\mathcal{F}(m+1-(i+1))) = 0$ for $i > 0$, then $H^i(\mathcal{H}(m-i)) = 0$, and hence assertion (ii). \square

Theorem 4.7 ([Mum66, Theorem in Lecture 14]). *For every numerical polynomial $p(t) \in \mathbb{Q}[t]$ there exists an integer m such that all coherent sheaves of ideals on \mathbb{P}_k^n having Hilbert polynomial $p(t)$ are m -regular.*

Proposition 4.8. *Let $m' \in \mathbb{N}$, and let $p(t) \in \mathbb{Q}[t]$ be a numerical polynomial. Then there exists an integer $m = m(m', p) \geq m'$, such that for every short exact sequence*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0 \quad (3)$$

of coherent sheaves on \mathbb{P}_k^n with \mathcal{G} being m' -regular and \mathcal{H} having Hilbert polynomial $p(t)$, the sheaf \mathcal{F} is m -regular.

Proof. Since m -regularity does not depend on the field k , we can without loss of generality assume that k is infinite.

We prove the proposition by induction on n . For $n = 0$, there is nothing to show, and we can directly assume that $n > 0$. Since k is infinite, there exists a hyperplane $j: H \subseteq \mathbb{P}_k^n$ such that in the diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}(-1) & \longrightarrow & \mathcal{F} & \longrightarrow & j_*(\mathcal{F}|_H) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{G}(-1) & \longrightarrow & \mathcal{G} & \longrightarrow & j_*(\mathcal{G}|_H) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{H}(-1) & \longrightarrow & \mathcal{H} & \longrightarrow & j_*(\mathcal{H}|_H) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array} \quad (4)$$

all sequences are exact. Indeed, take H so that it does not contain any of the associated points of \mathcal{F} , \mathcal{G} and \mathcal{H} . Then all horizontal and the first two vertical sequences are exact. Exactness of the third vertical sequence follows by diagram chase. Moreover, we get an induced short

exact sequence

$$0 \longrightarrow \mathcal{F}|_H \longrightarrow \mathcal{G}|_H \longrightarrow \mathcal{H}|_H \longrightarrow 0$$

on $H \cong \mathbb{P}_k^{n-1}$. Note that also $\mathcal{G}|_H$ is m' -regular by Lemma 4.6(ii). Furthermore, the Hilbert polynomial $q(t) = p(t) - p(t-1)$ of $\mathcal{H}|_H$ depends only on $p(t)$. By induction, there exists $m_1 = m_1(m', q) \in \mathbb{N}$ such that the restriction $\mathcal{F}|_H$ is m_1 -regular. This implies by [Kle71, Proposition 1.4] that \mathcal{F} is $(m_1 + h^1(\mathcal{F}(m_1 - 1)))$ -regular, and it remains to show that there exists an independent bound for $h^1(\mathcal{F}(m_1 - 1))$.

Note that by Proposition 4.4(i), we can assume that $m_1 \geq m'$. In particular, the sheaf \mathcal{G} is m_1 -regular, and hence $H^1(\mathcal{G}(m_1 - 1)) = 0$. Sequence (3) implies that the map $H^0(\mathcal{H}(m_1 - 1)) \rightarrow H^1(\mathcal{F}(m_1 - 1))$ is surjective, and hence $h^1(\mathcal{F}(m_1 - 1)) \leq h^0(\mathcal{H}(m_1 - 1))$. Finally, we claim that \mathcal{H} is m_1 -regular. This concludes the proof since then $h^0(\mathcal{H}(m_1 - 1)) = p(m_1 - 1)$, and we can set $m := m_1 + p(m_1 - 1)$.

To prove the claim, we first observe that, by Lemma 4.6(ii), it suffices to show that $H^i(\mathcal{F}(m_1 + 1 - i)) = 0$ for all $i > 1$. To simplify notation, we set $\mathcal{F}' := j_*(\mathcal{F}|_H)$. After twisting the short exact sequence

$$0 \longrightarrow \mathcal{F}(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow 0$$

from diagram (4) by $\mathcal{O}(d)$, we consider the induced long exact sequence

$$\dots H^{i-1}(\mathcal{F}'(d)) \longrightarrow H^i(\mathcal{F}(d-1)) \longrightarrow H^i(\mathcal{F}(d)) \longrightarrow H^i(\mathcal{F}'(d)) \dots$$

in cohomology. Let $i > 1$. Then we have $H^{i-1}(\mathcal{F}'(d)) = H^i(\mathcal{F}'(d)) = 0$ for $d \geq m_1 + 1 - i$ since the sheaf \mathcal{F}' is m_1 -regular. Therefore it follows that $H^i(\mathcal{F}(d-1)) \cong H^i(\mathcal{F}(d))$ for all $d \geq m_1 + 1 - i$. In particular, we see that $H^i(\mathcal{F}(m_1 - i)) \cong H^i(\mathcal{F}(m_1 - i + 1)) \cong H^i(\mathcal{F}(m_1 - i + 2)) \cong \dots$. But $H^i(\mathcal{F}(d)) = 0$ for $d \gg 0$, and we obtain $H^i(\mathcal{F}(m_1 + 1 - i)) = 0$. \square

4.2. Embedding over a field. The results on Castelnuovo–Mumford regularity can then be used to show that there exist integers m and N such that for every field k and every $(C, i) \in CM(\text{Spec}(k))$, the invertible sheaf $i^*\mathcal{O}_{\mathbb{P}_k^n}(m)$ is generated by $N + 1$ global sections and the corresponding map $j: C \rightarrow \mathbb{P}_k^n$ is a closed immersion.

Proposition 4.9. *There exists $m' \in \mathbb{N}$ such that for every field k and every pair $(C, i) \in CM(\text{Spec}(k))$ the coherent sheaf $i_*\mathcal{O}_C$ on \mathbb{P}_k^n is m' -regular.*

Proof. Consider the factorization

$$\begin{array}{ccc} C & \xrightarrow{i} & \mathbb{P}_k^n \\ & \searrow h & \nearrow j \\ & i(C) & \end{array}$$

through the scheme-theoretic image $i(C)$, where h is finite and an isomorphism away from finitely many closed points, and j is a closed immersion. Let \mathcal{J} and \mathcal{K} be the kernel and the cokernel of the natural map $\mathcal{O}_{\mathbb{P}_k^n} \rightarrow i_*\mathcal{O}_C$. Then we have short sequences

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_{\mathbb{P}_k^n} \longrightarrow j_*\mathcal{O}_{i(C)} \cong \mathcal{O}_{\mathbb{P}_k^n}/\mathcal{J} \longrightarrow 0 \quad (5)$$

and

$$0 \longrightarrow j_*\mathcal{O}_{i(C)} \longrightarrow i_*\mathcal{O}_C \longrightarrow \mathcal{K} \longrightarrow 0 \quad (6)$$

of $\mathcal{O}_{\mathbb{P}_k^n}$ -modules. We have seen in Proposition 2.14 that there are only finitely many possibilities for the Hilbert polynomial of $i(C)$. Consequently, by the exact sequence (5), there are only finitely many possibilities for the Hilbert polynomial of \mathcal{J} . By Theorem 4.7, there exists an integer m' independent of k , C and i such that \mathcal{J} is m' -regular. Consider sequences (5) and (6), and recall that $\mathcal{O}_{\mathbb{P}_k^n}$ and \mathcal{K} are 0-regular by Example 4.3 and 4.2 respectively. We see with Lemma 4.6 that also $i_*\mathcal{O}_C$ is m' -regular. \square

Proposition 4.10. *Let m' be as in Proposition 4.9. There exists an integer $m \geq m'$ such that for every field k , every $(C, i) \in CM(\text{Spec}(k))$ and every closed k -point $x \in C$, the sheaf $i_*\mathcal{J}$ is m -regular, where \mathcal{J} is the sheaf of ideals on C corresponding to the closed immersion $h: \text{Spec}(\kappa(x)) \hookrightarrow C$.*

Proof. Let k be a field, $(C, i) \in CM(\text{Spec}(k))$ and $x \in C$ be a k -rational point. The closed immersion h gives rise to a short exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_C \longrightarrow h_*\mathcal{O}_{\text{Spec}(\kappa(x))} \longrightarrow 0$$

of \mathcal{O}_C -modules. Since i is finite and hence in particular affine, also the induced sequence

$$0 \longrightarrow i_*\mathcal{J} \longrightarrow i_*\mathcal{O}_C \longrightarrow i_*h_*\mathcal{O}_{\text{Spec}(\kappa(x))} \longrightarrow 0$$

on \mathbb{P}_k^n is exact. Let $m' \in \mathbb{N}$ be as in Proposition 4.9. Then the coherent sheaf $i_*\mathcal{O}_C$ is m' -regular. Moreover, the coherent sheaf $i_*h_*\mathcal{O}_{\text{Spec}(\kappa(x))}$ has Hilbert polynomial constant equal to 1 since x is a k -rational point. Then the statement is a direct consequence of Proposition 4.8. \square

Lemma 4.11. *Let $f: X \rightarrow Y$ be an affine morphism of schemes, and let \mathcal{F} be a quasi-coherent sheaf on X . Suppose that the direct image $f_*\mathcal{F}$ is generated by its global sections. Then \mathcal{F} is generated by its global sections.*

Proof. Since $f_*\mathcal{F}$ is generated by its global sections, there exists a surjective map $\bigoplus_{i \in I} \mathcal{O}_Y \twoheadrightarrow f_*\mathcal{F}$. Applying the right exact inverse image functor f^* , we get a surjective map $\bigoplus_{i \in I} \mathcal{O}_X \twoheadrightarrow f^*f_*\mathcal{F}$. Since f is affine, the natural homomorphism $f^*f_*\mathcal{F} \rightarrow \mathcal{F}$ is surjective, see [Gro61a, Remarques (3.4.7)]. Composition of these two maps yields the required surjection. \square

Proposition 4.12. *Let m be as in Proposition 4.10. Let k be a field, and let $(C, i) \in CM(\text{Spec}(k))$.*

- (i) *The coherent sheaf $i^*\mathcal{O}_{\mathbb{P}_k^n}(m)$ is generated by its global sections, its higher cohomology vanishes and $h^0(i^*\mathcal{O}_{\mathbb{P}_k^n}(m)) = p(m)$.*
- (ii) *Let $x \in C$ be a k -rational point, and let \mathcal{J} be the sheaf of ideals of the closed immersion $h: \text{Spec}(\kappa(x)) \hookrightarrow C$. Then the coherent sheaf $\mathcal{J} \otimes_{\mathcal{O}_C} i^*\mathcal{O}_{\mathbb{P}_k^n}(m)$ is generated by its global sections and its higher cohomology vanishes.*

Proof. Let \mathcal{F} denote the structure sheaf \mathcal{O}_C or the sheaf of ideals \mathcal{J} . Note that in both cases the direct image $i_*\mathcal{F}$ is m -regular by the choice of m . Hence, by Proposition 4.4 and Corollary 4.5, the twist $(i_*\mathcal{F})(m)$ is generated by its global sections and its higher cohomology vanishes. Note further that $(i_*\mathcal{F})(m) \cong i_*(\mathcal{F} \otimes_{\mathcal{O}_C} i^*\mathcal{O}_{\mathbb{P}_k^n}(m))$ by the projection formula. Thus the proposition follows from Lemma 4.11 and the fact that we have $H^r(C, \mathcal{G}) = H^r(\mathbb{P}_k^n, i_*\mathcal{G})$ for every coherent sheaf \mathcal{G} on C and $r \geq 0$ since the morphism i is affine. \square

Lemma 4.13. *Let k be an algebraically closed field, and let X be a scheme over k . Let \mathcal{L} be an invertible sheaf on X that is generated by its global sections. Suppose further that $\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{L}$ is generated by its global sections for every sheaf of ideals \mathcal{J} corresponding to a closed point of X . Then the global sections of \mathcal{L} separate points and tangent vectors.*

Proof. Let x be a closed point of X , and let \mathcal{J} be the sheaf of ideals of the closed immersion $\text{Spec}(\kappa(x)) \hookrightarrow X$.

Let first $y \neq x$ be another closed point of X . Since the sheaf $\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{L}$ is globally generated, there exists a section $s \in H^0(X, \mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{L})$ such that $s_y \notin \mathfrak{m}_y(\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{L})_y = \mathfrak{m}_y\mathcal{L}_y$. Note that $s_x \in (\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{L})_x = \mathfrak{m}_x\mathcal{L}_x$. From the inclusion $H^0(X, \mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{L}) \subseteq H^0(X, \mathcal{L})$, we get that the global sections of \mathcal{L} separate points.

It remains to show that the vector space $\mathfrak{m}_x\mathcal{L}_x/\mathfrak{m}_x^2\mathcal{L}_x$ is spanned by the set $\{s \in H^0(X, \mathcal{L}) \mid s_x \in \mathfrak{m}_x\mathcal{L}_x\}$. Since $\mathfrak{m}_x\mathcal{L}_x = (\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{L})_x$, this is a direct consequence of the sheaf $\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{L}$ being generated by its global sections. \square

Proposition 4.14. *Let k be a field, and let $(C, i) \in CM(\text{Spec}(k))$. Let m be as in Proposition 4.10, and set $N := p(m) - 1$. Then the sheaf $i^*\mathcal{O}_{\mathbb{P}_k^n}(m)$ is very ample. In particular, there exists a closed immersion $j: C \hookrightarrow \mathbb{P}_k^N$ such that $j^*\mathcal{O}_{\mathbb{P}_k^N}(1) = i^*\mathcal{O}_{\mathbb{P}_k^n}(m)$.*

Proof. By Proposition 4.12, the invertible sheaf $\mathcal{L} := i^*\mathcal{O}_{\mathbb{P}_k^n}(m)$ is generated by its global sections and $h^0(C, \mathcal{L}) = p(m)$. Let s_0, \dots, s_N be a basis of $H^0(C, \mathcal{L})$. We claim that the map $j: C \rightarrow \mathbb{P}_k^N$ corresponding to \mathcal{L} and the sections s_0, \dots, s_N is a closed immersion. Note that j is a closed immersion if and only if the induced map $\bar{j}: C_{\bar{k}} \rightarrow \mathbb{P}_{\bar{k}}^N$, obtained

by base change to the algebraic closure \bar{k} of k , is a closed immersion. As we moreover have that $H^0(C, \mathcal{L}) \otimes_k \bar{k} = H^0(C_{\bar{k}}, \mathcal{L}_{\bar{k}})$, we can without loss of generality assume that k is algebraically closed.

The invertible sheaf \mathcal{L} satisfies the conditions of Lemma 4.13 by choice of m , and hence its global sections separate points and tangent vectors. Now it follows from [Har77, Proposition II.7.3] that \bar{j} is a closed immersion. \square

4.3. Embedding over a general base scheme. Now let S be a locally Noetherian scheme, and let $N \in \mathbb{N}$ be as in the previous subsection. We show that for every pair $(C, i) \in CM(S)$, the scheme S has an open cover $\{S_a\}$ such that for every a there exists a closed immersion $C_{S_a} \hookrightarrow \mathbb{P}_{S_a}^N$.

Theorem 4.15 (Cohomology and base change). *Let S be a locally Noetherian scheme, and let $f: X \rightarrow S$ be a proper morphism. Let \mathcal{F} be a coherent \mathcal{O}_X -module that is flat over S . Suppose that $H^r(X_s, \mathcal{F}_s) = 0$ for all $s \in S$ and $r > 0$. Then we have the following.*

- (i) *The \mathcal{O}_S -module $f_*\mathcal{F}$ is locally free.*
- (ii) *For every morphism $g: T \rightarrow S$ the natural base change map $g^*f_*\mathcal{F} \rightarrow (f_T)_*\mathcal{F}_T$ is an isomorphism.*

Proof. The first statement (i) is [Gro63, Corollaire (7.9.9)]. For statement (ii) consider [Gro63, Corollaire (6.9.9)] in the special case that $\mathcal{P}_0 = \mathcal{F}$ and $\mathcal{P}_i = 0$ for $i \neq 0$. \square

Proposition 4.16. *Let S be locally Noetherian. Let $(C, i) \in CM(S)$, and denote by $f: C \rightarrow S$ the structure map. Let further $\mathcal{L} := i^*\mathcal{O}_{\mathbb{P}_S^n}(m)$ with m as in Proposition 4.10. Then the following holds.*

- (i) *The sheaf $f_*\mathcal{L}$ is locally free of rank $p(m)$.*
- (ii) *For every morphism $g: T \rightarrow S$ the natural base change map $g^*f_*\mathcal{L} \rightarrow (f_T)_*\mathcal{L}_T$ is an isomorphism.*

Proof. For every $s \in S$, we have $\mathcal{L}_s = (i_s)^*\mathcal{O}_{\mathbb{P}_{\kappa(s)}^n}(m)$, and hence $H^r(C_s, \mathcal{L}_s) = 0$ for all $r > 0$ by Proposition 4.12. Then the statement follows directly from Theorem 4.15. \square

Proposition 4.17. *Let S be a locally Noetherian scheme, and let $(C, i) \in CM(S)$. With the notation as in Proposition 4.16, the natural map $\pi: f^*f_*\mathcal{L} \rightarrow \mathcal{L}$ is surjective.*

Proof. The cokernel \mathcal{N} of π is coherent, and it suffices to show that $\mathcal{N}_s = 0$ for every $s \in S$.

So let $s \in S$, and consider the fiber diagram

$$\begin{array}{ccc} C_s & \xrightarrow{h} & C \\ f_s \downarrow & & \downarrow f \\ \mathrm{Spec}(\kappa(s)) & \xrightarrow{g} & S. \end{array}$$

Since $\mathcal{L}_s = h^*\mathcal{L}$ is generated by its global sections by assumption on m , see Proposition 4.12, the natural map $\pi_s: f_s^*(f_s)_*\mathcal{L}_s \rightarrow \mathcal{L}_s$ is surjective. Note that the map $h^*\pi: h^*f^*f_*\mathcal{L} \rightarrow h^*\mathcal{L}$ factors as

$$h^*f^*f_*\mathcal{L} \cong f_s^*g^*f_*\mathcal{L} \xrightarrow{\alpha} f_s^*(f_s)_*h^*\mathcal{L} \xrightarrow{\pi_s} h^*\mathcal{L},$$

where α is the pullback of the natural map $g^*f_*\mathcal{L} \rightarrow (f_s)_*h^*\mathcal{L}$ which is an isomorphism by Proposition 4.16. It follows that also $h^*\pi$ is surjective, that is, $\mathcal{N}_s = 0$. \square

Proposition 4.18 ([Gro61b, Proposition (4.6.7)]). *Let S be a locally Noetherian scheme, and let $f: X \rightarrow Y$ be a morphism of proper S -schemes. Suppose that $f_s: X_s \rightarrow Y_s$ is finite (resp. a closed immersion) for a point $s \in S$. Then there exists an open neighborhood $U \subseteq S$ of s such that the restriction $f_U: X_U \rightarrow Y_U$ is finite (resp. a closed immersion).*

Proposition 4.19. *Let $(C, i) \in CM(S)$, where S is a locally Noetherian scheme. Let m be as in Proposition 4.10, and set $\mathcal{L} := i^*\mathcal{O}_{\mathbb{P}_S^n}(m)$. Suppose that $f_*\mathcal{L}$, where $f: C \rightarrow S$ denotes the structure map, is a free \mathcal{O}_S -module of rank $N + 1$. Then every isomorphism $\sigma: \mathcal{O}_S^{N+1} \xrightarrow{\sim} f_*\mathcal{L}$ gives rise to a closed immersion $j_\sigma: C \hookrightarrow \mathbb{P}_S^N$ with $j_\sigma^*\mathcal{O}_{\mathbb{P}_S^N}(1) = \mathcal{L}$.*

Proof. Composing the pull-back $f^*\sigma$ with the map $f^*f_*\mathcal{L} \rightarrow \mathcal{L}$, that is surjective by Proposition 4.17, gives rise to a S -morphism $j_\sigma: C \rightarrow \mathbb{P}_S^N$ such that $j_\sigma^*\mathcal{O}_{\mathbb{P}_S^N}(1) = \mathcal{L}$. We observe that j_σ is a closed immersion over every fiber by Proposition 4.14. Then it follows from Proposition 4.18 that j_σ itself is a closed immersion. \square

Theorem 4.20. *Let m be as in Proposition 4.10 and set $N := p(m) - 1$. For any locally Noetherian scheme S , and any $(C, i) \in CM(S)$, the invertible sheaf $\mathcal{L} := i^*\mathcal{O}_{\mathbb{P}_S^n}(m)$ is very ample for the structure morphism $f: C \rightarrow S$.*

Moreover, the scheme S has an open cover $\{S_a\}_{a \in I}$ such that for every $a \in I$ there exists a closed immersion $j_a: C_a := C \times_S S_a \hookrightarrow \mathbb{P}_{S_a}^N$ with $j_a^\mathcal{O}_{\mathbb{P}_{S_a}^N}(1) = i_a^*\mathcal{O}_{\mathbb{P}_{S_a}^n}(m)$.*

Proof. By Proposition 4.16, the coherent sheaf $f_*\mathcal{L}$ is locally free of rank $N + 1$. Let $\{S_a\}_{a \in I}$ be an open cover of S such that every restriction $(f_*\mathcal{L})|_{S_a}$ is free. For every $a \in I$ consider the induced element (C_a, i_a) in $CM(S_a)$ with structure map $f_a: C_a \rightarrow S_a$. According to Proposition 4.19, every choice of basis for $(f_*\mathcal{L})|_{S_a} = (f_a)_*i_a^*\mathcal{O}_{\mathbb{P}_{S_a}^n}(m)$ then gives a closed immersion $j_a: C_a \hookrightarrow \mathbb{P}_{S_a}^N$ with the required properties. \square

5. A COVERING OF CM

Let m be as in Proposition 4.10, and set $N := p(m) - 1$. We show in Subsection 5.1 that there exists a scheme W_0 parameterizing all closed

subschemes $Z \subseteq \mathbb{P}^n \times \mathbb{P}^N$ such that (Z, pr_1) is an element of CM , and the second projection $\text{pr}_2: Z \rightarrow \mathbb{P}^N$ is a closed immersion. Here pr_1 and pr_2 denote the projections $Z \rightarrow \mathbb{P}^n$ and $Z \rightarrow \mathbb{P}^N$ respectively. In Subsection 5.3, we construct a refinement W of W_0 corresponding to the closed subschemes $Z \subseteq \mathbb{P}^n \times \mathbb{P}^N$ as above such that the second projection pr_2 is given by the invertible sheaf $\text{pr}_1^* \mathcal{O}_{\mathbb{P}^n}(m)$. Moreover, we get a surjective map $\pi: W \rightarrow CM$ that maps Z to (Z, pr_1) .

5.1. Subscheme of the Hilbert scheme. Let $P(t) = p((m+1)t)$, and let $H := \text{Hilb}_{\mathbb{P}^n \times \mathbb{P}^N}^{P(t)}$ be the Hilbert scheme parameterizing closed subschemes of $\mathbb{P}^n \times \mathbb{P}^N$ having Hilbert polynomial $P(t)$ with respect to the very ample sheaf $\mathcal{O}(1, 1)$. In the following, we show that the functor $F: (\mathbf{Sch})^\circ \rightarrow \mathbf{Sets}$ given by

$$F(S) := \left\{ Z \subseteq \mathbb{P}_S^n \times_S \mathbb{P}_S^N \text{ in } H(S) \left| \begin{array}{l} (Z, \text{pr}_1) \in CM(S) \text{ and} \\ \text{pr}_2 \text{ is a closed immersion} \end{array} \right. \right\},$$

where $\text{pr}_1: Z \rightarrow \mathbb{P}_S^n$ and $\text{pr}_2: Z \rightarrow \mathbb{P}_S^N$ denote the projections, is represented by an open subscheme W_0 of H .

5.1.1. Properties of the projections.

Proposition 5.1. *Let S be locally Noetherian, and let $f: X \rightarrow Y$ be a morphism of proper S -schemes. There exists an open subscheme U of S such that a morphism $g: T \rightarrow S$ factors through U if and only if the morphism $f_T: X_T \rightarrow Y_T$ obtained by base change is finite (resp. a closed immersion).*

Proof. Let \mathcal{P} denote one of the properties “is a closed immersion” and “is finite”.

By Proposition 4.18, the set $U := \{s \in S \mid f_s: X_s \rightarrow Y_s \text{ has } \mathcal{P}\}$ is open, and the restriction $f_U: X_U \rightarrow Y_U$ has property \mathcal{P} . Since property \mathcal{P} is stable under base change, a morphism $T \rightarrow S$ that factors through U has \mathcal{P} .

Now consider a morphism $g: T \rightarrow S$ such that the induced map f_T has property \mathcal{P} . Then $(f_T)_t: (X_T)_t \rightarrow (Y_T)_t$ has \mathcal{P} for every $t \in T$. As the morphism $f_{g(t)}: X_{g(t)} \rightarrow Y_{g(t)}$ has \mathcal{P} if and only if the morphism $(f_T)_t$ obtained by the change of base field $\text{Spec}(\kappa(t)) \rightarrow \text{Spec}(\kappa(g(t)))$ has \mathcal{P} , it follows that $g(t) \in U$. Since U is an open subscheme, we get that g factors through U . \square

5.1.2. Cohen–Macaulay and of pure dimension 1.

Proposition 5.2. *Let $f: X \rightarrow S$ be a flat, proper morphism of finite presentation. Then there exists an open subscheme U of S such that a morphism $g: T \rightarrow S$ factors through U if and only if all fibers of X_T over T are Cohen–Macaulay and of pure dimension 1.*

Proof. Note that a fiber X_s for $s \in S$ is Cohen–Macaulay and of pure dimension 1 if and only if the structure sheaf $\mathcal{O}_{X_s} = (\mathcal{O}_X)_s$ does not have any embedded point and all irreducible components of $\text{Supp}((\mathcal{O}_X)_s)$ have dimension 1. By [Gro66, Théorème (12.2.1)(iv)], the set $U := \{s \in S \mid X_s \text{ is Cohen–Macaulay and of pure dimension 1}\}$ is then open.

Consider a morphism $g: T \rightarrow S$, and let $t \in T$. Then $(X_T)_t$ is obtained from $X_{g(t)}$ by base change $\text{Spec}(\kappa(t)) \rightarrow \text{Spec}(\kappa(g(t)))$. By Lemma 2.4, the fiber $(X_T)_t$ is Cohen–Macaulay and of pure dimension 1 if and only if $g(t) \in U$. The statement now follows as U is open. \square

5.1.3. Isomorphism onto image away from finite set of closed points.

Proposition 5.3. *Let $h: X \rightarrow Y$ be a finite morphism of schemes over a locally Noetherian scheme S . Suppose that Y is proper over S . Then there exists an open subscheme U of S such that a morphism $T \rightarrow S$ factors through U if and only if for every $t \in T$ the map $h_t: X_t \rightarrow Y_t$ is an isomorphism onto its image away from finitely many closed points.*

Proof. Consider the closed subset $Z = \text{Supp}(\text{coker}(\mathcal{O}_Y \rightarrow h_*\mathcal{O}_X))$ of Y with the induced reduced scheme structure. Then Chevalley’s upper semicontinuity theorem [Gro66, Théorème (13.1.5)] implies that the locus $U := \{s \in S \mid \dim(Z_s) = 0\}$ is open.

By Lemma 2.4(iii), a morphism $g: T \rightarrow S$ factors through the open subscheme U if and only if $\dim((Z_{g(t)})_t) = 0$ for every $t \in T$. As, moreover, $\dim((Z_{g(t)})_t) = \dim(\text{Supp}(\text{coker}(\mathcal{O}_{Y_T} \rightarrow (h_T)_*\mathcal{O}_{X_T}))_t)$, the statement follows with Lemma 2.3. \square

5.1.4. The Hilbert polynomial.

Proposition 5.4. *Let $h: X \rightarrow \mathbb{P}_S^n$ be a finite morphism of schemes over a locally Noetherian scheme S . Suppose that X is flat over S . Let further $p(x) \in \mathbb{Q}[x]$ be a numerical polynomial. Then there exists an open and closed subscheme U of S such that a morphism $g: T \rightarrow S$ factors through U if and only if the coherent sheaf $((h_T)_*\mathcal{O}_{X_T})_t$ has Hilbert polynomial $p(x)$ for every $t \in T$.*

Proof. Note that since X is flat over S and h is affine, the direct image $h_*\mathcal{O}_X$ is flat over S . In particular, its Hilbert polynomial is locally constant on S by [Gro63, Proposition (7.9.11)], and therefore the subset $U := \{s \in S \mid (h_*\mathcal{O}_X)_s \text{ has Hilbert polynomial } p(x)\}$ is open and closed. Moreover, we have that $(h_T)_*\mathcal{O}_{X_T} = (h_*\mathcal{O}_X)_T$ since h is affine. It follows that set U has the required property because the Hilbert polynomial is independent of the base field. \square

5.2. The covering scheme. We use the results of the previous subsection to construct a surjective map $W_0 \rightarrow CM$ from a scheme W_0 .

Theorem 5.5. *There exists a scheme W_0 of finite type over $\mathrm{Spec}(\mathbb{Z})$ such that*

$$W_0(S) = \left\{ Z \subseteq \mathbb{P}_S^n \times_S \mathbb{P}_S^N \mid \begin{array}{l} (Z, \mathrm{pr}_1) \in CM(S) \text{ and} \\ \mathrm{pr}_2: Z \rightarrow \mathbb{P}_S^N \text{ is a closed immersion} \end{array} \right\}$$

for all schemes S .

The map $\tau: W_0 \rightarrow CM$, given by mapping $Z \subseteq \mathbb{P}_S^n \times_S \mathbb{P}_S^N$ in $W_0(S)$ to (Z, pr_1) in $CM(S)$ for all S , is surjective as a map of Zariski sheaves.

Proof. Let H be the Hilbert scheme parameterizing closed subschemes of the product $\mathbb{P}^n \times \mathbb{P}^N$ with Hilbert polynomial $P(t) := p((m+1)t)$ with respect to the very ample sheaf $\mathcal{O}(1, 1) := \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^N}(1)$. Applying Propositions 5.1, 5.2, 5.3 and 5.4 to the universal family $\mathcal{Z}_H \subseteq \mathbb{P}_H^n \times_H \mathbb{P}_H^N$ over H and the projections $\mathcal{Z}_H \rightarrow \mathbb{P}_H^n$ and $\mathcal{Z}_H \rightarrow \mathbb{P}_H^N$, we see that all conditions in W_0 are open on H .

For surjectivity of τ , let S be a scheme and $(C, i) \in CM(S)$. By Theorem 4.20, we have an open covering $\{S_a\}$ of S and closed immersions $j_a: C_a \hookrightarrow \mathbb{P}_{S_a}^N$ such that $j_a^* \mathcal{O}_{\mathbb{P}_{S_a}^N}(1) = i^* \mathcal{O}_{\mathbb{P}_{S_a}^n}(m)$. For every a , the induced closed immersion $(i_a, j_a): C_{S_a} \hookrightarrow \mathbb{P}_{S_a}^n \times_{S_a} \mathbb{P}_{S_a}^N$ satisfies that $(i_a, j_a)^* \mathcal{O}(1, 1) = i_a^* \mathcal{O}_{\mathbb{P}_{S_a}^n}(m+1)$, and hence gives rise to an element of $W_0(S_a)$. \square

5.3. Refinement of the covering scheme.

Lemma 5.6. *Let S be a locally Noetherian scheme, and let \mathcal{E} and \mathcal{F} be finite locally free \mathcal{O}_S -modules. Then the functor $F: (\mathrm{Sch}_S)^\circ \rightarrow \mathbf{Sets}$ defined by*

$$F(T) = \{\mathcal{O}_T\text{-module isomorphisms } g^* \mathcal{E} \rightarrow g^* \mathcal{F}\},$$

where $g: T \rightarrow S$ denotes the structure map, is represented by a scheme of finite type over S .

Proof. We note first that F is an open subfunctor of $H: (\mathrm{Sch}_S)^\circ \rightarrow \mathbf{Sets}$ defined by $H(T) = \mathrm{Hom}_{\mathcal{O}_T}(g^* \mathcal{E}, g^* \mathcal{F})$. Now the statement follows as H is represented by the scheme $\mathbf{Spec}(\mathrm{Sym}(\mathcal{E} \otimes \mathcal{F}^\vee))$ that is of finite type over S . \square

Proposition 5.7. *Over the scheme W_0 of Theorem 5.5, consider the universal family $\mathcal{Z}_0 \subseteq \mathbb{P}_{W_0}^n \times_{W_0} \mathbb{P}_{W_0}^N$ with structure map $f: \mathcal{Z}_0 \rightarrow W_0$. Let $\mathcal{L} := \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}_{W_0}^n}(m)$, where m is as in Proposition 4.10, and set $\mathcal{E} := f_* \mathcal{L}$. Let $F: (\mathrm{Sch}_{W_0})^\circ \rightarrow \mathbf{Sets}$ be the functor defined by*

$$F(T) = \{\mathcal{O}_T\text{-module isomorphisms } \mathcal{O}_T^{N+1} \rightarrow g^* \mathcal{E}\},$$

for every W_0 -scheme $g: T \rightarrow W_0$. Then F is represented by a scheme W_1 that is of finite type over W_0 .

Proof. We have that $(\mathcal{Z}_0, \mathrm{pr}_1) \in CM(W_0)$. Hence the sheaf \mathcal{E} is locally free of rank $N+1$ by Proposition 4.16. Then the statement is a special case of Lemma 5.6. \square

Remark 5.8. Let T be a scheme. Then a morphism $h: T \rightarrow W_1$ corresponds to a closed subscheme $Z \subseteq \mathbb{P}_T^n \times \mathbb{P}_T^N$ and an isomorphism $\sigma: \mathcal{O}_T^{N+1} \xrightarrow{\sim} f_* \text{pr}_1^* \mathcal{O}_{\mathbb{P}_T^n}(m)$ of \mathcal{O}_T -modules, where $f: Z \rightarrow T$ denotes the structure map. By Proposition 4.19, the isomorphism σ gives rise to a closed immersion $j_h := j_\sigma: Z \hookrightarrow \mathbb{P}_T^N$.

Recall that, by definition of W_1 , the second projection $\text{pr}_2: Z \rightarrow \mathbb{P}_T^N$ is a closed immersion. In the following, we show that we can restrict ourselves to isomorphisms σ such that $j_\sigma = \text{pr}_2$. Explicitly, we prove that there exists a closed subscheme W of W_1 such that a morphism $h: T \rightarrow W_1$ factors through W if and only if $j_h = \text{pr}_2$.

Lemma 5.9. *Let $f, g: X \rightarrow Y$ be two morphisms over a scheme S , and suppose that Y is separated over S . Let $X_0 := X \times_{Y \times_S Y} Y$ be the fiber product with respect to the morphism $(f, g): X \rightarrow Y \times_S Y$ and the diagonal map $\Delta := (\text{id}, \text{id}): Y \rightarrow Y \times_S Y$, and denote the projections by $p: X_0 \rightarrow X$ and $p': X_0 \rightarrow Y$. Then $f = g$ if and only if the first projection p is an isomorphism.*

Proof. Being the base change of the closed immersion Δ , the morphism p is a closed immersion.

Suppose first that $f = g$. As then $\Delta \circ f = (f, f) = (f, g) \circ \text{id}_X$, by the universal property of the fiber product, there exists a map $q: X \rightarrow X_0$ such that $p \circ q = \text{id}_X$, see the following commutative diagram

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \searrow q & & \downarrow p' \\
 & X_0 & \xrightarrow{p'} Y \\
 \downarrow \text{id}_X & & \downarrow \Delta \\
 X & \xrightarrow{(f, g)} & Y \times_S Y
 \end{array}$$

It follows that the closed immersion p is an isomorphism.

Now assume conversely that p is an isomorphism, and let q be its inverse. Then we have $\text{pr}_i \circ (f, g) \circ p \circ q = \text{pr}_i \circ \Delta \circ p' \circ q$ for both projections $\text{pr}_1, \text{pr}_2: Y \times_S Y \rightarrow Y$. In particular, we get that $f = p' \circ q$ and $g = p' \circ q$, and hence $f = g$. \square

Proposition 5.10. *Let S be a Noetherian scheme, and let $f, g: X \rightarrow Y$ be morphisms of S -schemes, where X is projective and flat over S and Y is separated over S . Suppose that all fibers of X over S have Hilbert polynomial $p(t)$ with respect to some very ample sheaf \mathcal{L} on X . Then there exists a closed subscheme S' of S such that a map $h: T \rightarrow S$ factors through S' if and only if the morphisms $f_T: X_T \rightarrow Y_T$ and $g_T: X_T \rightarrow Y_T$ coincide.*

Proof. Consider the fiber product X_0 of $(f, g): X \rightarrow Y \times_S Y$ and the diagonal $\Delta: Y \rightarrow Y \times_S Y$. Note that Δ is a closed immersion since

Y is separated over S , and therefore we can consider X_0 as a closed subscheme of X . The construction of X_0 commutes with base change and hence, by Lemma 5.9, $f_T = g_T$ for a morphism $T \rightarrow S$ if and only if the induced closed immersion $(X_0)_T \hookrightarrow X_T$ is an isomorphism.

We consider the natural transformation $\mathcal{Hilb}_{X_0/S}^p \rightarrow \mathcal{Hilb}_{X/S}^p$ given by considering a closed subscheme of $(X_0)_T$ as closed subscheme of X_T via the inclusion $(X_0)_T \hookrightarrow X_T$. This natural transformation is a monomorphism. Note that since X_0 and X are projective over S , the functors $\mathcal{Hilb}_{X_0/S}^p$ and $\mathcal{Hilb}_{X/S}^p$ are represented by schemes $\text{Hilb}_{X_0/S}^p$ and $\text{Hilb}_{X/S}^p$ that are projective over S . In particular, we have a commutative diagram

$$\begin{array}{ccc} \text{Hilb}_{X_0/S}^p & \xrightarrow{j} & \text{Hilb}_{X/S}^p \\ & \searrow \pi_0 & \downarrow \pi \\ & & S \end{array}$$

with projective, and in particular proper and separated, maps π_0 and π . Then it follows from [Har77, Corollary 4.8] that the monomorphism j is proper. By [Gro67, Corrolaire (18.12.6)], this implies that j is a closed immersion.

The scheme X as a closed subscheme of itself corresponds to an element of $\mathcal{Hilb}_{X/S}^p(S)$, and hence to a map $S \rightarrow \text{Hilb}_{X/S}^p$. Let S' be the fiber product of S and $\text{Hilb}_{X_0/S}^p$ over $\text{Hilb}_{X/S}^p$. Then S' is a closed subscheme of S . Moreover, from the commutativity of the fiber diagram

$$\begin{array}{ccc} S' & \xrightarrow{\quad} & S \\ \downarrow & & \downarrow \\ \text{Hilb}_{X_0/S}^p & \longrightarrow & \text{Hilb}_{X/S}^p, \end{array}$$

we get that $(X_0)_{S'} = X_{S'}$. Suppose conversely that $h: T \rightarrow S$ is a morphism such that $(X_0)_T \hookrightarrow X_T$ is an isomorphism. Then the composition $T \rightarrow S \rightarrow \text{Hilb}_{X/S}^p$ corresponds to the closed subscheme X_T of X_T . Since $(X_0)_T = X_T$, the map $T \rightarrow \text{Hilb}_{X/S}^p$ factors through $\text{Hilb}_{X_0/S}^p$. It follows that $T \rightarrow S$ factors through the fiber product S' , and hence S' has the required properties. \square

Proposition 5.11. *Let W_1 be the W_0 -scheme of Proposition 5.7. With the notation as in Remark 5.8, there exists a closed subscheme W of W_1 such that a morphism $h: T \rightarrow W_1$ factors through W if and only if $j_h = \text{pr}_2$.*

Proof. Over W_1 we have the universal family $\mathcal{Z}_1 \subseteq \mathbb{P}_{W_1}^n \times_{W_1} \mathbb{P}_{W_1}^N$ with induced closed immersion $j := j_{\text{id}}: \mathcal{Z}_1 \hookrightarrow \mathbb{P}_{W_1}^N$. As j_h is the base change of j by h , the statement follows directly from Proposition 5.10. \square

Let $\pi: W \rightarrow CM$ be the composition $W \hookrightarrow W_1 \rightarrow W_0 \xrightarrow{\tau} CM$.

Theorem 5.12. *The map $\pi: W \rightarrow CM$ is surjective as a map of Zariski sheaves.*

In particular, the map $\pi(\mathrm{Spec}(k)): W(\mathrm{Spec}(k)) \rightarrow CM(\mathrm{Spec}(k))$ is surjective for every field k .

Proof. Let S be a scheme, and let $(C, i) \in CM(S)$ with structure map $f: C \rightarrow S$. By Theorem 4.20, there exists an open cover $\{S_a\}$ of S , such that the sheaf $f_* i^* \mathcal{O}_{\mathbb{P}_S^n}(m)$ is free of rank $N + 1$ over every S_a , and a choice of basis of global sections gives rise to a closed immersion $j_a: C_a \hookrightarrow \mathbb{P}_{S_a}^N$ with $j_a^* \mathcal{O}_{\mathbb{P}_{S_a}^N}(1) = i_a^* \mathcal{O}_{\mathbb{P}_S^n}(m)$. Then the closed immersion $(i_a, j_a): C_a \rightarrow \mathbb{P}_{S_a}^n \times_{S_a} \mathbb{P}_{S_a}^N$ corresponds to an element of $W_0(S_a)$ mapping to $(C_a, i_a) \in CM(S_a)$. The choice of the basis gives a factorization of the corresponding map $S_a \rightarrow W_0$ through W_1 . Moreover, by construction of j_a , it even factors through W . \square

Corollary 5.13. *For every scheme S and map $S \rightarrow CM$, the induced map $\pi_S: W \times_{CM} S \rightarrow S$ of schemes is surjective, that is, the map $\pi: W \rightarrow CM$ is surjective as representable map.*

Proof. First we note that the statement makes sense since π is representable by Corollary 3.6.

Let $S \rightarrow CM$ be a map from a scheme S , and consider a map $\mathrm{Spec}(k) \rightarrow S$ for a field k . Composition then gives an element of $CM(\mathrm{Spec}(k))$ having a lift in $W(\mathrm{Spec}(k))$ by Theorem 5.12. Hence the original map $\mathrm{Spec}(k) \rightarrow S$ factors through the fiber product $W \times_{CM} S$. By [GW10, Proposition 4.8], this implies that π_S is a surjective map of schemes. \square

6. SMOOTHNESS OF THE COVERING

Let $\pi: W \rightarrow CM$ be the surjective map of Theorem 5.12. The goal of this section is to show that π is smooth. Moreover, we conclude that CM is an algebraic space.

Lemma 6.1. *Let A be a ring, and let I be a nilpotent ideal of A .*

- (i) *Let $\varphi: M \rightarrow N$ be a homomorphism of A -modules, and let $\bar{\varphi}: M/IM \rightarrow N/IN$ be the induced map. Suppose that N is flat over A . Then φ is an isomorphism if and only if $\bar{\varphi}$ is an isomorphism.*
- (ii) *Let N be a flat A -module, and let $\bar{\varphi}: (A/I)^n \rightarrow N/IN$ be an isomorphism. Then there exists an isomorphism $\varphi: A^n \rightarrow N$ such that $\varphi \otimes \mathrm{id}_{A/I} = \bar{\varphi}$.*

Proof. Clearly $\bar{\varphi}$ is an isomorphism if φ is an isomorphism.

Suppose conversely that $\bar{\varphi}$ is an isomorphism, and let C be the cokernel of φ . Then we have that $C/IC = \mathrm{coker}(\bar{\varphi}) = 0$. It follows that

$C = IC = I^2C = \dots = 0$ since I is nilpotent. This shows that φ is surjective.

Now let $K := \ker(\varphi)$, and consider the short exact sequence

$$0 \longrightarrow K \longrightarrow M \xrightarrow{\varphi} N \longrightarrow 0.$$

Since N is flat, the sequence remains exact after tensoring with A/I . In particular, we get that $K/IK = \ker(\bar{\varphi}) = 0$. As for the cokernel, this implies that $K = 0$ and assertion (i) follows.

To show assertion (ii), we lift the basis of N/IN given by $\bar{\varphi}$ to N , and get a well-defined map $\varphi: A^n \rightarrow N$ that is an isomorphism after tensoring with A/I . Now it follows from assertion (i) that φ is an isomorphism. \square

Theorem 6.2. *The map $\pi: W \rightarrow CM$ is formally smooth, that is, for every ring A with nilpotent ideal I and all morphism $\text{Spec}(A/I) \rightarrow W$ and $\text{Spec}(A) \rightarrow CM$ such that the diagram*

$$\begin{array}{ccc} \text{Spec}(A/I) & \longrightarrow & W \\ \downarrow & \nearrow & \downarrow \pi \\ \text{Spec}(A) & \longrightarrow & CM \end{array} \quad (7)$$

commutes, there exists at least one morphism $\text{Spec}(A) \dashrightarrow W$ making the diagram commute.

Proof. Consider a commutative diagram as in the statement of the theorem. The map $\text{Spec}(A) \rightarrow CM$ corresponds to an element (C, i) in $CM(\text{Spec}(A))$ with structure morphism $f: C \rightarrow \text{Spec}(A)$. Base change by the closed immersion $\text{Spec}(A/I) \hookrightarrow \text{Spec}(A)$ gives rise to an element $(\bar{C}, \bar{i}) \in CM(\text{Spec}(A/I))$ with structure morphism $\bar{f}: \bar{C} \rightarrow \text{Spec}(A/I)$, where we write $\bar{C} := C_{A/I}$.

The map $\text{Spec}(A/I) \rightarrow W$ corresponds to a closed subscheme \bar{Z} of the product $\mathbb{P}_{A/I}^n \times_{A/I} \mathbb{P}_{A/I}^N$ with structure map $\bar{g}: \bar{Z} \rightarrow \text{Spec}(A/I)$ and an isomorphism $\bar{\sigma}: \mathcal{O}_{\text{Spec}(A/I)}^{N+1} \xrightarrow{\sim} \bar{g}_* \text{pr}_1^* \mathcal{O}_{\mathbb{P}_{A/I}^n}(m)$ such that $j_{\bar{\sigma}} = \text{pr}_2$.

Note that $(\bar{Z}, \text{pr}_1) = (\bar{C}, \bar{i})$ in $CM(\text{Spec}(A/I))$ since diagram (7) commutes. Hence there is an isomorphism $\alpha: \bar{C} \xrightarrow{\sim} \bar{Z}$ with $\text{pr}_1 \circ \alpha = \bar{i}$. Composing the induced isomorphism $\bar{g}_* \text{pr}_1^* \mathcal{O}_{\mathbb{P}_{A/I}^n}(m) \cong \bar{f}_* \bar{i}^* \mathcal{O}_{\mathbb{P}_{A/I}^n}(m)$ with $\bar{\sigma}$ gives an isomorphism $\bar{\rho}: \mathcal{O}_{\text{Spec}(A/I)}^{N+1} \xrightarrow{\sim} \bar{f}_* \bar{i}^* \mathcal{O}_{\mathbb{P}_{A/I}^n}(m)$. Then the closed immersion $j_{\bar{\rho}}$ associated to $\bar{\rho}$ equals the composed map $\text{pr}_2 \circ \alpha: \bar{C} \hookrightarrow \mathbb{P}_{A/I}^N$.

Using the identification $(f_* i^* \mathcal{O}_{\mathbb{P}_A^n}(m))_{A/I} \cong \bar{f}_* \bar{i}^* \mathcal{O}_{\mathbb{P}_{A/I}^n}(m)$ of Proposition 4.16, we can lift $\bar{\rho}$ to an isomorphism $\rho: \mathcal{O}_A^{N+1} \xrightarrow{\sim} f_* i^* \mathcal{O}_{\mathbb{P}_A^n}(m)$ by Lemma 6.1(ii). By Proposition 4.19, we get an induced closed immersion $j_{\rho}: C \hookrightarrow \mathbb{P}_A^N$.

The combined closed immersion $(i, j_{\rho}): C \hookrightarrow \mathbb{P}_A^n \times_A \mathbb{P}_A^N$ gives rise to a map $\gamma: \text{Spec}(A) \rightarrow W$. By construction, the base change of j_{ρ} by

the closed immersion $\mathrm{Spec}(A/I) \hookrightarrow \mathrm{Spec}(A)$ is the map $\mathrm{pr}_2 \circ \alpha = j_{\bar{\rho}}$. It follows that the constructed map γ makes the diagram (7) commute. \square

Remark 6.3. Note that the map γ constructed in the proof is not unique. It depends on the choice of a lift ρ of $\bar{\rho}$. In particular, the map $\pi: W \rightarrow CM$ is not formally étale. However, the existence of a smooth cover suffices to show that CM is an algebraic space.

Proposition 6.4. *Let $\mathcal{A}: \mathbf{Sch}^\circ \rightarrow \mathbf{Sets}$ be a functor that is a sheaf in the étale topology. Suppose that there exists a scheme X with a representable, surjective and smooth map $X \rightarrow \mathcal{A}$. Then \mathcal{A} is an algebraic space.*

Proof. See [LMB00, Chapter 8] for the construction of an étale cover of \mathcal{A} . \square

Theorem 6.5. *The functor CM is an algebraic space of finite type over $\mathrm{Spec}(\mathbb{Z})$.*

Proof. We have seen in Theorem 2.26 that CM is a sheaf in the étale topology. Moreover, we have a scheme W of finite type over $\mathrm{Spec}(\mathbb{Z})$ and a map $\pi: W \rightarrow CM$ that is representable by Proposition 3.4, surjective by Theorem 5.12, locally of finite presentation by Corollary 3.6 and formally smooth by Theorem 6.2. Now it follows from Proposition 6.4 that CM is an algebraic space. \square

7. PROPERNESS OF CM

We conclude by showing that CM satisfies the valuative criterion for properness.

7.1. Flat families over a discrete valuation ring. Flatness over a discrete valuation ring can be described in terms of associated points.

Lemma 7.1 ([Mum66, Proposition 6 in Lecture 6]). *Let R be a discrete valuation ring, and let X be a scheme over $\mathrm{Spec}(R)$ with structure morphism $f: X \rightarrow \mathrm{Spec}(R)$. A coherent sheaf \mathcal{F} on X is flat over $\mathrm{Spec}(R)$ if and only if every associated point of \mathcal{F} is contained in the generic fiber X_η , where η denotes the generic point of $\mathrm{Spec}(R)$.*

Proposition 7.2. *Let R be a discrete valuation ring, and let η be the generic point of $\mathrm{Spec}(R)$. Let $f: X \rightarrow \mathrm{Spec}(R)$ be a flat and proper morphism, and suppose that the generic fiber X_η has pure dimension n . Then X has pure dimension $n + 1$. Moreover, the closed fiber has pure dimension n .*

Proof. Let X' be an irreducible component of X . Since f is flat and proper, the generic point is mapped to η and the closed points in X' are mapped to the closed point in $\mathrm{Spec}(R)$. Hence the restriction of

f to X' is surjective, and it follows from [GW10, Proposition 14.107] that $\dim(X') = \dim(\operatorname{Spec}(R)) + \dim((X')_\eta) = 1 + n$.

By [Gro66, Corollaire (14.2.4)], the set of dimensions of the irreducible components of the closed fiber is contained in the set of dimensions of the irreducible components of the generic fiber. This implies the statement on the closed fiber. \square

Corollary 7.3. *Let R be a discrete valuation ring, and let X be a scheme that is flat and proper over $\operatorname{Spec}(R)$. Suppose that the generic fiber X_η has pure dimension 1. Let x be a closed point of X . Then x has codimension 2 in every irreducible component of X containing it. In particular, we have $\dim(\mathcal{O}_{X,x}) = 2$.*

Proof. By Proposition 7.2, the scheme X has pure dimension 2. Let $x \in X$ be a closed point, and let X' be an irreducible component of X containing it. Note that X' is flat and proper over $\operatorname{Spec}(R)$. In particular, the closed fiber $(X')_0$ has pure dimension 1 by Proposition 7.2, it contains the closed point x , and we have $\operatorname{codim}(\{x\}, (X')_0) = 1$. Moreover, the closed fiber $(X')_0$ has codimension 1 in X' by *Krull's Hauptidealsatz*. This shows that x has codimension at least 2 in X' . \square

7.2. Family of Cohen–Macaulay curves over a discrete valuation ring. We fix the following notation. Let R be a discrete valuation ring with maximal ideal $\mathfrak{m} = (\pi)$ and field of fractions $K = \operatorname{Quot}(R)$.

For a family of Cohen–Macaulay curves $(C, i) \in \operatorname{CM}(\operatorname{Spec}(R))$, we denote the scheme-theoretic image $i(C)$ of i by Z . Then the morphism i factors as

$$\begin{array}{ccc} C & \xrightarrow{i} & \mathbb{P}_R^n \\ & \searrow h & \nearrow \\ & Z & \end{array}$$

By construction, the associated map $h^\# : \mathcal{O}_Z \rightarrow h_* \mathcal{O}_C$ is injective, and we set $Y := \operatorname{Supp}(\operatorname{coker}(h^\#))$ for the locus of points in Z where C and Z are not isomorphic.

Further we write (C_K, i_K) in $\operatorname{CM}(\operatorname{Spec}(K))$ for the restriction of (C, i) to the generic point in $\operatorname{Spec}(R)$. Then the generic fiber Z_K of Z is the scheme-theoretic image $i_K(C_K)$ of C_K in \mathbb{P}_K^n . In particular, with the induced map $h_K : C_K \rightarrow Z_K$, we have that $h_K^\# : \mathcal{O}_{Z_K} \rightarrow (h_K)_* \mathcal{O}_{C_K}$ is injective, and we set $Y_K := \operatorname{Supp}(\operatorname{coker}(h_K^\#))$. Note that Y_K is the restriction of Y to the generic fiber.

7.2.1. Scheme-theoretic image. First we study the scheme-theoretic image Z . We show that it is the scheme-theoretic closure of its generic fiber Z_K in \mathbb{P}_R^n , and that it has pure dimension 2.

Proposition 7.4. *The scheme-theoretic image Z of C in \mathbb{P}_R^n is the scheme-theoretic closure of Z_K in \mathbb{P}_R^n . In particular, Z is flat over*

$\mathrm{Spec}(R)$, it has pure dimension 2 and $\dim(\mathcal{O}_{Z,z}) = 2$ for every closed point z in Z .

Proof. The scheme C is flat over $\mathrm{Spec}(R)$, and hence its generic fiber C_K is scheme-theoretically dense in C . Thus the scheme Z is the scheme-theoretic image of the composition $C_K \hookrightarrow C \rightarrow \mathbb{P}_R^n$. Moreover, by definition of Z_K , the scheme-theoretic closure of Z_K in Z equals the scheme-theoretic image of the composition $C_K \rightarrow Z_K \hookrightarrow \mathbb{P}_K^n \rightarrow \mathbb{P}_R^n$. Now the first statement follows as the diagram

$$\begin{array}{ccc} C_K & \xrightarrow{i_K} & \mathbb{P}_K^n \\ \downarrow & & \downarrow \\ C & \xrightarrow{i} & \mathbb{P}_R^n \end{array}$$

commutes. Moreover, the associated points of Z lie in the generic fiber Z_K , and hence Z is flat over $\mathrm{Spec}(R)$ by Lemma 7.1.

Note that Z_K has pure dimension 1 by Proposition 2.12. Then the remaining statements follow directly from Proposition 7.2 and Corollary 7.3. \square

Lemma 7.5. *Let $f: X \rightarrow Y$ be a finite morphism of locally Noetherian schemes. A point $x \in X$ is closed if and only if its image $f(x)$ is a closed point of Y .*

Proof. Since the finite morphism f is closed, the image of a closed point is closed. Suppose conversely that the image $f(x)$ is a closed point. By [Gro61a, Corollaire (6.1.7)], the morphism f is quasi-finite. In particular, the fiber $f^{-1}(f(x))$ is finite and closed, that is, it consists of finitely many closed points. \square

Proposition 7.6. *Let $(C, i) \in \mathrm{CM}(\mathrm{Spec}(R))$ for a discrete valuation ring R , and let $h: C \rightarrow Z$ be the induced map to the image. Then $\dim(\mathcal{O}_{C,x}) = \dim(\mathcal{O}_{Z,h(x)})$ for every point $x \in C$.*

Proof. After passing to suitable affine neighborhoods of x and $h(x)$, the *Cohen–Seidenberg Theorem* [Mat80, (5.E) Theorem 5] gives that $\dim(\mathcal{O}_{Z,h(x)}) \geq \dim(\mathcal{O}_{C,x})$ for every $x \in C$. We have to show that the inequality cannot be strict.

Since both C and Z have pure dimension 2, Lemma 7.5 implies that $\dim(\mathcal{O}_{Z,h(x)}) = 2$ if and only if $\dim(\mathcal{O}_{C,x}) = 2$. It remains to eliminate the possibility that $\dim(\mathcal{O}_{Z,h(x)}) = 1$ and $\dim(\mathcal{O}_{C,x}) = 0$. If x is a generic point of C , then it lies in the generic fiber C_K . Consequently, the image $h(x)$ lies in the generic fiber Z_K that, by Corollary 2.13, has pure dimension 1. It follows that $\dim(\mathcal{O}_{Z,h(x)}) = \dim(\mathcal{O}_{Z_K,h(x)}) \leq 1$. As the point $h(x)$ is not closed in Z_K by Lemma 7.5, we conclude that $\dim(\mathcal{O}_{Z,h(x)}) = 0$. \square

Corollary 7.7. *Let $(C, i) \in CM(\text{Spec}(R))$ be a family of Cohen–Macaulay curves over a discrete valuation ring R . Let $x_1, x_2 \in C$ be two points in C such that $h(x_1) = h(x_2)$, where $h: C \rightarrow Z$ is the induced map onto the image. Then $\dim(\mathcal{O}_{C, x_1}) = \dim(\mathcal{O}_{C, x_2})$.*

Proof. This follows directly from Proposition 7.6. \square

7.2.2. *Cohen–Macaulayness of the direct image.* Next, we show that the direct image sheaf $h_*\mathcal{O}_C$ is Cohen–Macaulay as a \mathcal{O}_Z -module.

Lemma 7.8 ([Gro65, Corollaire (5.7.11)]). *Let $\varphi: A \rightarrow B$ be a homomorphism of Noetherian rings, and let M be a module over B that is finitely generated as an A -module.*

- (i) *Suppose that M is Cohen–Macaulay as an A -module. Then M is Cohen–Macaulay as a B -module.*
- (ii) *Suppose that $\dim_{B_{\mathfrak{q}_1}}(M_{\mathfrak{q}_1}) = \dim_{B_{\mathfrak{q}_2}}(M_{\mathfrak{q}_2})$ for all prime ideals $\mathfrak{q}_1, \mathfrak{q}_2 \in \text{Spec}(B)$ such that $\varphi^{-1}(\mathfrak{q}_1) = \varphi^{-1}(\mathfrak{q}_2)$. If M is Cohen–Macaulay as a B -module, then M is Cohen–Macaulay as a module over A .*

Proposition 7.9. *The direct image sheaf $h_*\mathcal{O}_C$ is Cohen–Macaulay as an \mathcal{O}_Z -module.*

Proof. Note that since both $\text{Spec}(R)$ and all fibers of C over $\text{Spec}(R)$ are Cohen–Macaulay, it follows from Proposition 2.9 that C is Cohen–Macaulay. In particular, the sheaf \mathcal{O}_C is Cohen–Macaulay as an \mathcal{O}_C -module.

By Lemma 7.8(ii), it suffices to show that $\dim(\mathcal{O}_{C, x_1}) = \dim(\mathcal{O}_{C, x_2})$ for all points $x_1, x_2 \in C$ such that $h(x_1) = h(x_2)$. Then Corollary 7.7 concludes the proof. \square

7.2.3. *The non-isomorphism locus.* In the next step, we show that the locus where C and Z are not isomorphic is the union of the non-Cohen–Macaulay locus in Z and the closure of the locus where the generic fibers C_K and Z_K are not isomorphic.

Recall that we defined $Y = \text{Supp}(\text{coker}(h^\#))$, for the natural inclusion $h^\#: \mathcal{O}_Z \hookrightarrow h_*\mathcal{O}_C$. Then $Y_K = \text{Supp}(\text{coker}((h_K)^\#))$ is the intersection of Y with the generic fiber Z_K . Let Y_1 be the closure of the set Y_K in Z .

Lemma 7.10. *The set $Y_2 := \{z \in Z \mid \mathcal{O}_{Z, z} \text{ is not Cohen–Macaulay}\}$ is closed in Z .*

Proof. As Z is a closed subscheme of the regular scheme \mathbb{P}_R^n , the statement follows directly from [Gro65, Corollaire (6.11.3)]. \square

In Proposition 7.13, we show that these two sets Y_1 and Y_2 constitute the entire non-isomorphism locus. In particular, this implies that Y is uniquely determined by the generic fiber (C_K, i_K) .

The following results will be useful, see also [Kol09, Lemma 36]. We use the notation $\text{depth}_Y(\mathcal{F}) := \inf_{y \in Y} \text{depth}_{\mathcal{O}_{X,y}}(\mathcal{F}_y)$ for a closed subset Y of a locally Noetherian scheme X and a coherent \mathcal{O}_X -module \mathcal{F} .

Proposition 7.11. *Let X be a locally Noetherian scheme, and let $\alpha: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of \mathcal{O}_X -modules where \mathcal{F} is coherent and \mathcal{G} is quasi-coherent.*

- (i) *Suppose that there exists an open subscheme U of X such that the restriction of α to U is injective. If $\text{depth}_{X \setminus U}(\mathcal{F}) \geq 1$, then α is injective.*
- (ii) *Suppose that there exists an open subscheme U of X such that the restriction of α to U is an isomorphism. If $\text{Ass}(\mathcal{G}) \subseteq U$ and $\text{depth}_{X \setminus U}(\mathcal{F}) \geq 2$, then α is an isomorphism.*

Proof. Since all properties are local on X , we can without loss of generality assume that $X = \text{Spec}(A)$ for a Noetherian ring A . Then we have $X \setminus U = V(I)$ for an ideal I of A , and $\mathcal{F} = \tilde{M}$ and $\mathcal{G} = \tilde{N}$ for A -modules M and N . Note that

$$\text{depth}_{X \setminus U}(\mathcal{F}) = \inf_{\mathfrak{p} \supseteq I} \text{depth}(M_{\mathfrak{p}}) = \text{grade}(I, M)$$

by [BH93, Proposition 1.2.10].

Suppose first that $\alpha|_U$ is injective and $\text{depth}_{X \setminus U}(\mathcal{F}) \geq 1$. Then there exists an element $a \in I$ that is not a zero divisor of M . Let $K := \ker(\alpha) \subset M$, and take $m \in K$. Since the open subset $D(a)$ is contained in U , the restriction of α to $D(a)$ is injective, and hence $K_a = 0$. Therefore there exists some $n \in \mathbb{N}$ such that $a^n m = 0$. But a is not a zero divisor of M , so $m = 0$ and α is injective.

To show that the conditions in assertion (ii) moreover imply that α is surjective, suppose that $\alpha(M) \subsetneq N$. Let (a, b) be a M -regular sequence in I . In particular, the element a is not a zero divisor of M . Moreover, since $\text{Ass}(N) = \text{Ass}(M) \cap U$, it follows that a is not a zero divisor of N . The restriction of α to both open subsets $D(a), D(b) \subseteq U$ is surjective, so there exists an element $n \in N \setminus \alpha(M)$ such that $an, bn \in \alpha(M)$, say $an = \alpha(m_1)$ and $bn = \alpha(m_2)$ for $m_1, m_2 \in M$. Since α is injective, it follows that $bm_1 = am_2$. In particular, the element bm_1 lies in aM , and thus, as the sequence (a, b) is M -regular, there exists an element $m'_1 \in M$ such that $m_1 = am'_1$. We get that $an = \alpha(m_1) = a\alpha(m'_1)$. Since a is not a zero divisor in N , this implies that $n = \alpha(m'_1) \in \alpha(M)$, a contradiction. \square

Corollary 7.12. *Let X be a locally Noetherian scheme, and let \mathcal{F} be a coherent sheaf on X . Let $j: U \hookrightarrow X$ be an open subscheme of X such that $\text{depth}_{X \setminus U}(\mathcal{F}) \geq 2$. Then $\mathcal{F} \cong j_*(\mathcal{F}|_U)$.*

Proof. Consider the natural map $\alpha: \mathcal{F} \rightarrow j_*(\mathcal{F}|_U)$ that is the identity when restricted to U . Since $\text{Ass}(j_*(\mathcal{F}|_U)) = \text{Ass}(\mathcal{F}|_U) \subseteq U$ by [Gro65,

Proposition (3.1.13)], it follows from Proposition 7.11(ii) that α is an isomorphism. \square

Proposition 7.13. *In the situation of Subsection 7.2.3, the non-isomorphism locus is the union of the closure Y_1 of the non-isomorphism locus of the generic fiber and the non-Cohen–Macaulay locus Y_2 , that is, $Y = Y_1 \cup Y_2$.*

Proof. The non-isomorphism locus Y_K of the generic fiber is contained in the closed set Y and hence even its closure Y_1 in Z . Moreover, since the sheaf $h_*\mathcal{O}_C$ is Cohen–Macaulay as an \mathcal{O}_Z -module by Proposition 7.9, the non-Cohen–Macaulay locus Y_2 of Z is contained in Y . This shows the inclusion $Y \supseteq Y_1 \cup Y_2$.

Suppose that the inclusion is strict, that is, $Y' := Y_1 \cup Y_2 \subsetneq Y$. Since $Y \cap Z_K = Y_K = Y' \cap Z_K$, the complement $Y \setminus Y'$ lies over the closed point of $\text{Spec}(R)$. As the closed fiber of Y has dimension 0 by Lemma 2.3, it follows that $Y \setminus Y'$ consists of closed points that have codimension 2 in Z by Proposition 7.4. Consider the scheme $X := Z \setminus Y'$ and its open subscheme $U := Z \setminus Y$. Let further $\alpha: \mathcal{O}_X \hookrightarrow (h_*\mathcal{O}_C)|_X$ be the restriction of $h^\#$ to X . By definition of U , the restriction of α to U is an isomorphism. The coherent sheaf $(h_*\mathcal{O}_C)|_X$ is Cohen–Macaulay by Proposition 7.9, and hence $\text{depth}_{X \setminus U}((h_*\mathcal{O}_C)|_X) = 2$. Finally we observe that all associated points of \mathcal{O}_Z lie in the generic fiber, and in particular in U . Then it follows from Proposition 7.11(ii) that α is an isomorphism, contradicting the assumption. We conclude that $Y = Y_1 \cup Y_2$ as claimed. \square

Proposition 7.14. *Consider the open subscheme $U := Z_K \cup (Z \setminus Y)$ of Z , and let $j: U \hookrightarrow Z$ be the open immersion. The \mathcal{O}_Z -algebras $h_*\mathcal{O}_C$ and $j_*((h_*\mathcal{O}_C)|_U)$ are isomorphic.*

Proof. The complement $Z \setminus U = Y \setminus Y_K$ is contained in the zero-dimensional closed fiber of Y . In particular, every point in $Z \setminus U$ has codimension 2 in Z by Proposition 7.4. As $h_*\mathcal{O}_C$ is Cohen–Macaulay by Proposition 7.9, we consequently have that $\text{depth}_{Z \setminus U}(h_*\mathcal{O}_C) = 2$. Then $h_*\mathcal{O}_C \cong j_*((h_*\mathcal{O}_C)|_U)$ by Corollary 7.12. The isomorphism there is an isomorphism of \mathcal{O}_Z -algebras. \square

7.3. Valutive criterion for separatedness. The results of the previous section particularly imply that every $(C, i) \in CM(\text{Spec}(R))$ is uniquely determined by its generic fiber (C_K, i_K) , that is, the functor CM satisfies the valutive criterion for separatedness.

Theorem 7.15 (Valutive criterion for separatedness). *Let R be a discrete valuation ring with field of fractions K . For every commutative*

diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{\quad} & CM \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ \mathrm{Spec}(R) & \xrightarrow{\quad} & \mathrm{Spec}(\mathbb{Z}) \end{array}$$

there exists at most one map $\mathrm{Spec}(R) \dashrightarrow CM$ making the diagram commute. In other words, every element $(C, i) \in CM(\mathrm{Spec}(R))$ is uniquely determined by its generic fiber $(C_K, i_K) \in CM(\mathrm{Spec}(K))$.

Proof. Let (C, i) and (C', i') be two elements in $CM(\mathrm{Spec}(R))$ having the same generic fiber $(C_K, i_K) = (C'_K, i'_K)$ in $CM(\mathrm{Spec}(K))$. Then there exists an isomorphism $\alpha_K: C_K \xrightarrow{\sim} C'_K$ with $i'_K \circ \alpha_K = i_K$. In particular, C_K and C'_K have the same scheme-theoretic image Z_K in \mathbb{P}_K^n and the same non-isomorphism locus $Y_K \subset Z_K$. By Proposition 7.4, it follows that C and C' have the same scheme-theoretic image Z in \mathbb{P}_R^n , and we write $h: C \rightarrow Z$ and $h': C' \rightarrow Z$ for the restrictions of i and i' to the image. Then the isomorphism α_K induces an isomorphism $\beta_{Z_K}: (h_*\mathcal{O}_C)|_{Z_K} \xrightarrow{\sim} (h'_*\mathcal{O}_{C'})|_{Z_K}$ of \mathcal{O}_{Z_K} -algebras. By Proposition 7.13, both non-isomorphism loci $Y = \mathrm{Supp}(\mathrm{coker}(h^\#))$ and $Y' = \mathrm{Supp}(\mathrm{coker}((h')^\#))$ are given as the union of closure of the non-isomorphism locus Y_K of the generic fiber and non-Cohen–Macaulay locus of Z , and we have $Y = Y'$. Moreover, we get an induced $\mathcal{O}_{Z \setminus Y}$ -algebra isomorphism $\beta_{Z \setminus Y}: (h_*\mathcal{O}_C)|_{Z \setminus Y} \xrightarrow{\sim} (h'_*\mathcal{O}_{C'})|_{Z \setminus Y}$.

Note that the homomorphisms β_{Z_K} and $\beta_{Z \setminus Y}$ coincide on the intersection $Z_K \cap (Z \setminus Y) = Z_K \setminus Y_K$, and hence they glue to an isomorphism $\beta: (h_*\mathcal{O}_C)|_U \xrightarrow{\sim} (h'_*\mathcal{O}_{C'})|_U$ on the union $U := Z_K \cup (Z \setminus Y)$.

Let $j: U \hookrightarrow Z$ be the open immersion. Then we get an isomorphism $j_*\beta: j_*((h_*\mathcal{O}_C)|_U) \xrightarrow{\sim} j_*((h'_*\mathcal{O}_{C'})|_U)$ of \mathcal{O}_Z -algebras. By Proposition 7.14, this gives an isomorphism $h_*\mathcal{O}_C \xrightarrow{\sim} h'_*\mathcal{O}_{C'}$ of \mathcal{O}_Z -algebras, that is, an isomorphism $\alpha: C \xrightarrow{\sim} C'$ over Z , and hence over \mathbb{P}_R^n . This shows that $(C, i) = (C', i')$ in $CM(\mathrm{Spec}(R))$. \square

7.4. Valuative criterion for properness. The aim of this subsection is to show that any element in $CM(\mathrm{Spec}(K))$ can be lifted to $CM(\mathrm{Spec}(R))$.

So take $(C_K, i_K) \in CM(\mathrm{Spec}(K))$. Let $Z_K \subseteq \mathbb{P}_K^n$ be the scheme-theoretic image of i_K , and $h_K: C_K \rightarrow Z_K$ be the induced map to it. Recall that Z_K has pure dimension 1 by Corollary 2.13.

Let Z be the scheme-theoretic closure of Z_K in \mathbb{P}_R^n . Then Z is flat by Lemma 7.1, it has pure dimension 2 by Proposition 7.2 and all closed points have codimension 2 by Corollary 7.3. Let Y_1 be the closure of the non-isomorphism locus $Y_K := \mathrm{Supp}(\mathrm{coker}(\mathcal{O}_{Z_K} \rightarrow (h_K)_*\mathcal{O}_{C_K}))$ in Z and $Y_2 := \{z \in Z \mid \mathcal{O}_{Z,z} \text{ is not Cohen–Macaulay}\}$ be the non-Cohen–Macaulay locus of Z , and set $Y := Y_1 \cup Y_2$. Note that Y_2 , and hence also Y , is closed in Z by Lemma 7.10.

Proposition 7.16. *Consider the open subscheme $U := Z_K \cup (Z \setminus Y)$ of Z . Then $\text{codim}(Z \setminus U, Z) = 2$.*

Proof. Let $z \in Z \setminus U$. We have to show that $\dim(\mathcal{O}_{Z,z}) = 2$. Note that $Z \setminus U = (Y_1 \setminus Y_K) \cup Y_2$. Suppose first that $z \in Y_1 \setminus Y_K$. Then $z \in \overline{\{y\}}$ for a point $y \in Y_K$, and hence $\dim(\mathcal{O}_{Z,z}) > \dim(\mathcal{O}_{Z,y})$. Moreover, we have $\dim(\mathcal{O}_{Z,y}) = \dim(\mathcal{O}_{Z_K,y}) > 0$ since the point y is closed in the scheme Z_K of pure dimension 1. Next suppose that $z \in Y_2$. Then z lies in the closed fiber over $\text{Spec}(R)$ and, since Z is flat over $\text{Spec}(R)$, the local ring $\mathcal{O}_{Z,z}$ is a flat R -algebra. In particular, the generator π of the maximal ideal of R is not a zero divisor of $\mathcal{O}_{Z,z}$ and hence $\text{depth}_{\mathcal{O}_{Z,z}}(\mathcal{O}_{Z,z}) \geq 1$. But we have $\dim(\mathcal{O}_{Z,z}) > \text{depth}(\mathcal{O}_{Z,z})$ since \mathcal{O}_Z is not Cohen–Macaulay at the point z . \square

On $U_1 = Z_K$ we have the coherent \mathcal{O}_{U_1} -algebra $\mathcal{A}_1 := (h_K)_* \mathcal{O}_{C_K}$, and on $U_2 = Z \setminus Y$ we consider the coherent \mathcal{O}_{U_2} -algebra $\mathcal{A}_2 := \mathcal{O}_Z|_{U_2}$. Note that the intersection $U_1 \cap U_2 = Z_K \setminus Y_K$ is the locus where \mathcal{A}_1 and \mathcal{A}_2 coincide, and hence they glue to a coherent \mathcal{O}_U -algebra \mathcal{A}_U on the union $U = U_1 \cup U_2$.

Lemma 7.17. *The \mathcal{O}_U -module \mathcal{A}_U is Cohen–Macaulay.*

Proof. To show that the restriction $\mathcal{A}_1 = (h_K)_* \mathcal{O}_{C_K}$ of \mathcal{A}_U to U_1 is Cohen–Macaulay we can make use of Lemma 7.8. Then it suffices to show that $\dim(\mathcal{O}_{Z_K,x_1}) = \dim(\mathcal{O}_{Z_K,x_2})$ for all $x_1, x_2 \in Z_K$ with $h_K(x_1) = h_K(x_2)$. As Z_K has pure dimension 1, this follows from Lemma 7.5.

The restriction \mathcal{A}_2 to U_2 is Cohen–Macaulay since U_2 is obtained by removing the non-Cohen–Macaulay locus Y_2 of Z . \square

Let $j: U \hookrightarrow Z$ be the open immersion. We will show that the direct image $j_* \mathcal{A}_U$ gives rise to a lift C of the curve C_K . To see this, we need the following two results.

Proposition 7.18 ([Gro65, Corollaire (5.11.4)]). *Let X be a locally Noetherian scheme that can locally be embedded into a regular scheme. Let U be an open subset of X , and denote by $i: U \hookrightarrow X$ the inclusion. Let \mathcal{F} be a coherent \mathcal{O}_U -module. The direct image $i_* \mathcal{F}$ is coherent if and only if $\text{codim}((X \setminus U) \cap \overline{\{x\}}, \overline{\{x\}}) \geq 2$ for every $x \in \text{Ass}(\mathcal{F})$.*

Proposition 7.19 ([Gro65, Proposition (5.10.10)(i)]). *Let X be a locally Noetherian scheme, let U be an open subscheme of X , and denote by $i: U \hookrightarrow X$ the inclusion. Let \mathcal{F} be a coherent \mathcal{O}_U -module, and suppose that the direct image $i_* \mathcal{F}$ is coherent. Then $\text{depth}_{X \setminus U}(i_* \mathcal{F}) \geq 2$.*

Applying these results to the coherent \mathcal{O}_U -algebra \mathcal{A}_U that we defined above gives the following.

Proposition 7.20. *Let the notation be as above. The sheaf $\mathcal{A} := j_* \mathcal{A}_U$ of \mathcal{O}_Z -algebras is coherent and Cohen–Macaulay as an \mathcal{O}_Z -module.*

Proof. We have to show that \mathcal{A}_U satisfies the properties of Proposition 7.18. By Lemma 7.17, the sheaf \mathcal{A}_U is Cohen–Macaulay, and hence the associated points of \mathcal{A}_U are the generic points of $\text{Supp}(\mathcal{A}_U) = U$. Thus we have to show that $\text{codim}((Z \setminus U) \cap Z', Z') = 2$ for every irreducible component Z' of Z . By Proposition 7.16, the closed set $Z \setminus U$ has codimension 2 in Z , and hence it consists of closed points. We have seen in Corollary 7.3 that every closed point has codimension 2 in every irreducible component containing it. This shows that \mathcal{A} is coherent. Since $\text{depth}_{Z \setminus U} \mathcal{A} = 2$ by Proposition 7.19, it follows with Lemma 7.17 that \mathcal{A} is Cohen–Macaulay. \square

Now we are ready to prove that the moduli functor CM is proper.

Theorem 7.21 (Valuative criterion for properness). *Let R be a discrete valuation ring with field of fractions K . For every commutative diagram*

$$\begin{array}{ccc} \text{Spec}(K) & \longrightarrow & CM \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ \text{Spec}(R) & \longrightarrow & \text{Spec}(\mathbb{Z}) \end{array}$$

there exists a unique map $\text{Spec}(R) \dashrightarrow CM$ making the diagram commute. In other words, every pair $(C_K, i_K) \in CM(\text{Spec}(K))$ is the generic fiber of a unique element $(C, i) \in CM(\text{Spec}(R))$.

Proof. Let $(C_K, i_K) \in CM(\text{Spec}(K))$. The morphism i_K factors as $C_K \xrightarrow{h_K} Z_K \hookrightarrow \mathbb{P}_K^n$, where Z_K is the scheme-theoretic image of C_K in \mathbb{P}_K^n . Let Z be the scheme-theoretic closure of Z_K in \mathbb{P}_R^n . Let further Y_1 be the closure of $Y_K := \text{Supp}(\text{coker}(h_K^\#))$ in Z and Y_2 be the non-Cohen–Macaulay locus of Z . Let $Y := Y_1 \cup Y_2$, and let $U_1 := Z_K$, $U_2 := Z \setminus Y$ and $U := U_1 \cup U_2$. The algebras $(h_K)_* \mathcal{O}_{C_K}$ on U_1 and \mathcal{O}_{U_2} on U_2 coincide on the intersection $U_1 \cap U_2$, and glue to a coherent \mathcal{O}_U -algebra \mathcal{A}_U . We set $\mathcal{A} := j_*(\mathcal{A}_U)$, where $j: U \hookrightarrow Z$ denotes the inclusion. Let $C := \text{Spec}(\mathcal{A})$, and let $i: C \rightarrow \mathbb{P}_R^n$ be the composition of the structure map $C \rightarrow Z$ and the embedding $Z \hookrightarrow \mathbb{P}_R^n$. Note that base change to $\text{Spec}(K)$ gives the finite morphism $i_K: C_K \rightarrow \mathbb{P}_K^n$.

Finally, we claim that (C, i) defines an element in $CM(\text{Spec}(R))$. First we observe that, by Proposition 7.20, the \mathcal{O}_Z -module \mathcal{A} is coherent, and therefore the morphism i is finite. Moreover, as we have $\text{Ass}(\mathcal{A}) = \text{Ass}(\mathcal{A}_U)$ by [Gro65, Proposition (3.1.13)], all associated points of \mathcal{A} lie in the generic fiber over $\text{Spec}(R)$. By Lemma 7.1, it follows that C is flat over $\text{Spec}(R)$. It remains to show that the pair (C, i) satisfies the conditions on the fibers over $\text{Spec}(R)$. Note that restriction to the generic point gives the element (C_K, i_K) that lies in $CM(\text{Spec}(K))$ by assumption. Hence, we only have to check the closed fiber. It follows from Proposition 7.2 that also the closed fiber C_0 has pure dimension 1. Moreover, the sheaf \mathcal{A} is Cohen–Macaulay as an

\mathcal{O}_Z -module by Proposition 7.20, and hence also as an \mathcal{A} -module, see Lemma 7.8(i). It follows that the scheme C is Cohen–Macaulay. The closed fiber C_0 is obtained by dividing out with the non zero divisor π , and it is therefore Cohen–Macaulay by [BH93, Theorem 2.1.3(a)]. Finally, the closed fiber Y_0 of Y has dimension 0 by Proposition 7.16. By Lemma 2.3, this implies that also the restriction $i_0: C_0 \rightarrow \mathbb{P}_k^n$ is an isomorphism onto its image away from finitely many closed points. This concludes the proof since the Hilbert polynomial is locally constant. \square

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